Mean Curvature Flow with Free Boundary

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Outline

I. Review on Mean Curvature Flow

II. Mean Curvature Flow with boundary

III. A new convergence result

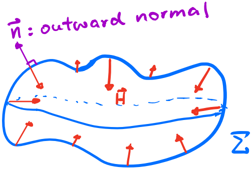
IV. Proof of the main result

I. Review on Mean Curvature Flow

Mean Curvature Flow

A family of hypersurfaces $\{\Sigma_t^n\}$ in $\mathbb{R}^{n+1}$ is said to satisfy the Mean Curvature Flow if they are moving with velocity equal to the mean curvature vector:

$$\partial_t \vec{x} = \vec{H} = -H\vec{n} \qquad \text{(MCF)}$$



- *Geometry:* MCF is the negative gradient flow for area.
- *Analysis:* MCF is a non-linear parabolic partial differential equation.
- *Physics:* Models for evolution of soap films, grain boundaries cf. **Mullins**.

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Existence and Uniqueness

By parabolic PDE theory, we have short-time existence and uniqueness:

Theorem (Hamilton (1982), Huisken (1984))

Given a compact smooth hypersurface $\Sigma \subset \mathbb{R}^{n+1}$, there exists a unique smooth maximal solution $\{\Sigma_t\}_{t\in[0,T)}$ to (MCF) with $\Sigma_0 = \Sigma$ such that

$$\max_{\Sigma_t} |A|^2 \to \infty \qquad \text{as } t \to T < +\infty.$$

Therefore, the flow will encounter singularities in finite time. This leads to

Two Fundamental Questions:

1. *What kind of singularities can occur?*
2. *How to continue the flow through singularities?*

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Huisken's Monotonicity Formula

Huisken (1990) proved the celebrated monotonicity formula (for $t < 0$)

$$\frac{d}{dt}\int_{\Sigma_t}\Phi = -\int_{\Sigma_t}\left|H\mathbf{n} + \frac{x^\perp}{2t}\right|^2 \leq 0 \qquad \text{where } \Phi(x,t) = (-4\pi t)^{-\frac{n}{2}} e^{\frac{|x|^2}{4t}}.$$

As a consequence, the singularities are modelled on *self-similar solutions*.

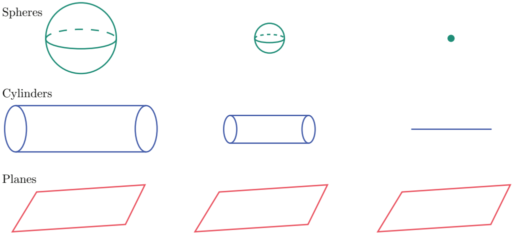

Figure 1. Cylinders, spheres, and planes are self-similar solutions of MCF. The shape is preserved, but the scale changes with time.

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Singularity Models in $\mathbb{R}^3$

There are 3 types of self-similar solutions: *shrinkers, translators* and *expanders*. Among them, the most important ones are shrinkers, which only undergo homothetic changes under the flow.

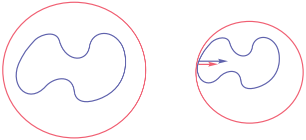
Other than round spheres and cylinders, **Angenent (1989)** constructed a shrinking donut. Numerical evidence by **Chopp (1994)** and **Ilmanen (1995)** suggests that many other examples exist. Some of these examples are constructed recently by gluing methods cf. **Nguyen (2014)**, **Kapouleas-Kleene-Möller (2018)**.

It seems out of reach to obtain a complete classification of singularity models. Instead, one may ask the following questions:

1. What are the *generic* singularities? c.f. **Colding-Minicozzi (2012)**
2. What if we impose further *geometric assumptions*?

Consequences of Maximum Principle

- *Avoidance Principle:* Two hypersurfaces that are initially disjoint remain disjoint. In particular, embeddedness is preserved under the flow. Moreover, compact MCF in $\mathbb{R}^{n+1}$ must become extinct in finite time.



- Let $\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_n$ be the principal curvatures of Σ_t, **Huisken (1984)** and **Huisken-Sinestrari (2009)** proved that the following conditions are preserved under the flow:
  1. *convex*: $\kappa_1 > 0$
  2. *two-convex*: $\kappa_1 + \kappa_2 > 0$
  3. *mean convex*: $H = \kappa_1 + \cdots + \kappa_n > 0$

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Contracting hypersurfaces to a point in $\mathbb{R}^{n+1}$

Under certain assumptions, only the singularity of shrinking spheres can occur.

Theorem (Huisken (1984))

Any compact convex hypersurface in $\mathbb{R}^{n+1}$ converges to a "round point".

When $n = 1$, **Gage-Hamilton (1986)** and **Grayson (1987)** showed that any *simple closed* curve in $\mathbb{R}^2$ converges to a round point. **Andrews-Bryan (2011)** gave a new proof using the two-point maximum principle.

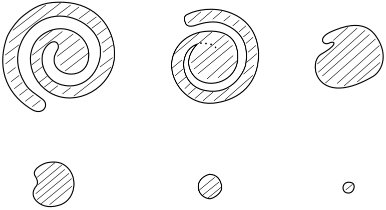

FIGURE 2. The snake manages to unwind quickly enough to become convex before extinction.

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MCF in Riemannian manifolds

Theorem (Huisken (1986))

Let (M^{n+1}, g) be a complete Riemannian manifold with positive injectivity radius inj $(M, g) \geq i_0 > 0$ that satisfies the following uniform curvature bounds:

$$-K_1 \leq K \leq K_2 \quad \text{and} \quad |\nabla Rm| \leq L,$$

Then, any initial hypersurface Σ_0 satisfying

$$H\, h_{ij} > nK_1\, g_{ij} + \frac{n^2}{H} L\, g_{ij}$$

would shrink to a round point in finite time under MCF.

In particular, any compact convex hypersurface in $\mathbb{S}^{n+1}$ converges to a round point in finite time.

When $n = 1$, **Grayson (1989)** showed a dichotomy that any simple closed curve in a closed surface (M^2, g) would either (i) converge to a round point in finite time or (ii) converge to a simple closed geodesic as $t \to T$.

Weak notions of MCF

There are several ways to continue the flow after singularities have occured.

1. *MCF with surgery*

   Idea: stop the flow very close to the first singular time, then remove regions of large curvature and replace by more regular ones cf. **Huisken-Sinestrari (2009)**, **Brendle-Huisken (2016)** and **Haslhofer-Kleiner (2017)**

2. *Level set flow*

   Idea: represent the evolving hypersurface as the level sets of a function $v(x,t)$ where $\Sigma_t = \{x \in \mathbb{R}^{n+1} \ : \ v(x,t) = 0\}$ cf. **Evans-Spruck (1991)**, **Chen-Giga-Goto (1991)**, **Colding-Minicozzi (2016-2019)**

3. *Brakke flow*

   Idea: use Geometric Measure Theory to define the flow of singular hypersurfaces with "good" compactness properties cf. **Brakke (1978)**, **Ilmanen (1994)**, **White (2000, 2002, 2015)**

Grayson's dumbbell

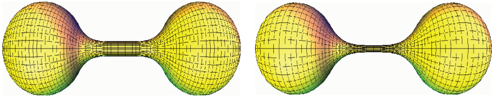

FIGURE 4. Grayson's dumbbell; initial surface and step 1.

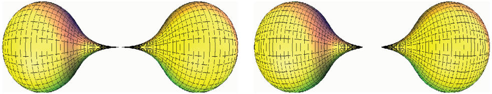

FIGURE 5. The dumbbell; steps 2 and 3.

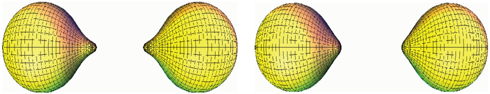

FIGURE 6. The dumbbell; steps 4 and 5.

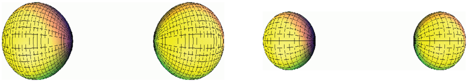

FIGURE 7. The dumbbell; steps 6 and 7 (see also [May]).

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II. Mean Curvature Flow with boundary

Mean Curvature Flows with boundary

Question

Can we evolve hypersurfaces *with boundary* under MCF?

YES, provided that suitable boundary conditions are imposed. Two types of commonly considered boundary conditions are:

1. *Dirichlet:* The motion of the boundary $\partial\Sigma_t$ is either fixed or prescribed cf. **White (1995, 2019)**
2. *Neumann:* The boundary $\partial\Sigma_t$ can move freely on a given hypersurface $S \subset \mathbb{R}^{n+1}$ and Σ_t is either orthogonal to S or with prescribed contact angle cf. **Huisken (1989)**, **Altschuler (1994)**, **Stahl (1996)**

Remark: The corresponding boundary value problems for Ricci flow is more subtle cf. **Gianniotis (2016)**

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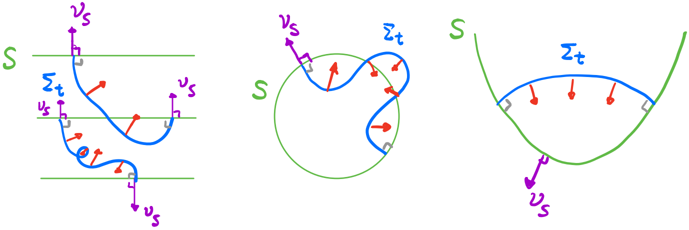
Free-boundary MCF

Definition

Let $S \subset \mathbb{R}^{n+1}$ be a smooth embedded hypersurface without boundary, oriented by the unit normal ν_S. A family $\{\Sigma_t\}$ of hypersurfaces with boundary is evolving by the free-boundary Mean Curvature Flow w.r.t. the "barrier" S if

1. Σ_t satisfies (MCF) in the interior
2. $\partial\Sigma_t \subset S$ and $\Sigma_t \perp S$ along $\partial\Sigma_t$ from "inside" of S



Some results on free-boundary MCF

Huisken (1989) obtained long time convergence of the flow for graphs over compact domain in $\mathbb{R}^n$. Various graphical settings are also considered by **Wheeler (2014, 2017)** and **Wheeler-Wheeler (2017)**.

Stahl (1996) established the short-time existence and uniqueness for compact initial data. **Buckland (2005)** proved a Huisken-type monotonicity formula. In the mean convex setting, **Edelen (2016)** showed the convexity estimates along the lines of **Huisken-Sinestrari (1999)**.

For weak solutions, the level set flow in the free boundary setting was first introduced by **Giga-Sato (1992)**. **Edelen (2018)** defined the corresponding notion of Brakke flow. **Mizuno-Tonegawa (2018)** and **Kagaya (2017)** etc. studied the Allen-Cahn equation counterpart. Recently, the regularity theory of White was generalized by **Edelen-Haslhofer-Ivaki-Zhu (2019)** to the free boundary setting.

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III. A new convergence result

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Convergence results for free-boundary MCF

Question

Under what conditions would a hypersurface converge to a "round half-point"?

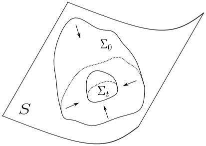

FIGURE 1. A convex surface with free boundary contained in a convex barrier surface is evolving under mean curvature flow to a shrinking hemisphere.

Theorem (Stahl (1996), Edelen (2016))

Any compact convex hypersurface in $\mathbb{R}^{n+1}$ with free boundary lying on $S = \mathbb{R}^n$ or $\mathbb{S}^n$ converges to a "round half-point" in finite time.

What about for other S?

A new convergence result for free-boundary MCF

In a joint work with Sven Hirsch, we generalize Stahl's convergence result to general convex barrier surfaces in $\mathbb{R}^3$.

Theorem (Hirsch-L. (2000) arXiv:2001.01111)

Let $S \subset \mathbb{R}^3$ be a smooth embedded oriented surface satisfying uniform bounds on the second fundamental form

$$|\nabla A_S| + |\nabla^2 A_S| \leq L$$

and bounds on the interior/exterior ball curvatures

$$0 \leq \underline{Z}_S \leq \overline{Z}_S \leq K_2.$$

Then, any compact surface which is "sufficiently convex", depending only on L and K_2, with free boundary lying on S will shrink to a round half-point in finite time under free-boundary MCF.

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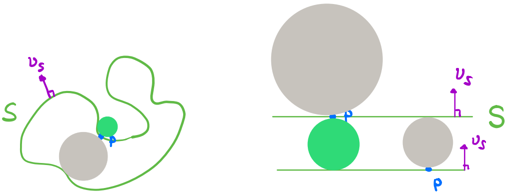
Interior/exterior ball curvatures

In studying non-collapsing of MCF, Andrews defined the interior ball curvature of S w.r.t. the outward normal ν_S at $p \in S$ by

$$\overline{Z}_S(p) := \sup_{p \neq q \in S} \left\{ \frac{2\langle p - q, \nu_S(p)\rangle}{|p-q|^2} \right\},$$

which is the curvature of the largest "interior" ball touching S at p.



The exterior ball curvature $\underline{Z}_S(p)$ is similarly defined with inf instead. The ball curvatures control both the principal curvatures and the inscribed radius of S:

- $\underline{Z}_S \geq 0 \Leftrightarrow S$ is convex
- $\overline{Z}_S(p) \geq \max \kappa_i(p)$

Huisken's convergence theorem revisited

Theorem (Huisken (1986))

Let (M^{n+1}, g) be a complete Riemannian manifold with positive injectivity radius inj $(M,g) \geq i_0 > 0$ that satisfies the following uniform curvature bounds:

$$-K_1 \leq K \leq K_2 \quad \text{and} \quad |\nabla Rm| \leq L,$$

Then, any initial hypersurface Σ_0 satisfying

$$H\, h_{ij} > nK_1\, g_{ij} + \frac{n^2}{H} L\, g_{ij} \qquad (*)$$

would shrink to a round point in finite time under MCF.

Comparing with our main theorem:

- we require up to second order derivative bounds on A_S
- the injectivity radius bound i_0 is replaced by the ball curvature bound K_2
- we do not have a sharp preserved inequality (*)

IV. Proof of the main result

Main difficulties and key ideas in the proof

Our proof follows the general strategy in **Huisken (1984)**. There are, however, several new features in our proof which do not appear in the boundaryless case:

1. possibility of boundary extrema
2. uncontrollable cross terms of second fundamental form at the boundary
3. loss of umbilicity (even if Σ_0 and S are totally umbilic)

We will deal with these new difficulties as follow:

- apply **Edelen (2016)**'s weight function trick to force the extrema away from the boundary
- introduce a new perturbation of the second fundamental form to get a reasonable boundary normal derivative
- establish new convexity and pinching estimates with "controlled decay"

Step 1: Finite extinction time

Claim: H_{min} blows up in finite time

The evolution equation for H reads

$$(\partial_t - \Delta) H = |A|^2 H$$

By maximum principle, any positive lower bound on H is preserved under the flow and thus H_{min} must blow up in finite time unless H_{min} occurs at a boundary point! However, this is impossible if S is convex since the boundary derivative of H satisfies

$$\frac{\partial}{\partial \eta} H = h^S_{\nu\nu} H \geq 0.$$

Step 2: Preserving convexity

Claim: For $D >> 1$, $h_{ij} \geq D g_{ij}$ at $t = 0 \Rightarrow h_{ij} \geq \frac{D}{3} g_{ij}$ for all t

The evolution equation for h_{ij} reads

$$(\partial_t - \Delta)\, h_{ij} = -2H h_{im} h^m{}_j + |A|^2 h_{ij}$$

By Hamilton's tensor maximum principle, any non-negative lower bound on h_{ij} is preserved under the flow unless the minimum occurs at a boundary point! The boundary normal derivatives are given by

$$\nabla_1 h_{11} = 2h^S_{22} H + (h^S_{\nu\nu} - 3h^S_{22}) h_{11} + \nabla^S_\nu h^S_{22} \quad (1)$$

$$\nabla_1 h_{22} = h^S_{22} H + (h^S_{\nu\nu} - 3h^S_{22}) h_{22} - \nabla^S_\nu h^S_{22} \quad (2)$$

Here, $\{e_1, e_2, \nu\}$ is an O.N.B. for $\mathbb{R}^3$ along $\partial\Sigma$ such that $e_1 = \nu_S$, $e_2 \in T(\partial\Sigma)$ and $\nu \perp \Sigma$. Notice that:

- The R.H.S. of (1) and (2) do not have a sign.
- $\nabla_1 h_{12}$ is not controllable by lower order terms. (Irrelevant for umbilic S!)

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Step 2: Preserving convexity (continued)

Idea: Introduce a perturbation term to h_{ij} so that the cross term vanishes.

Consider an auxiliary 5-tensor P on $\mathbb{R}^3$, depending only on S, defined by

$$\begin{aligned} P(U,V,X,Y,Z) :=& \left(A_S(U,X)\nu_S^\flat(V) + A_S(V,X)\nu_S^\flat(U)\right) g_S(Y,Z) \\ & - \left(g_S(U,X)\nu_S^\flat(V) + g_S(V,X)\nu_S^\flat(U)\right) A_S(Y,Z). \end{aligned}$$

We define the perturbed second fundamental form $\tilde{A}$ of Σ as

$$\tilde{A}(X,Y) := A(X,Y) + P^\Sigma(X,Y)$$

where $P^\Sigma(X,Y) := P(X,Y,\nu,\nu,\nu)$.We have along $\partial\Sigma$:

- $\tilde{h}_{11} = h_{11}$, $\tilde{h}_{22} = h_{22}$ and $\tilde{h}_{12} = 0$; (note that $h_{12} = -h^S_{2\nu}$) cf. **Edelen (2016)**
- $\nabla_1 \tilde{h}_{ij} = \nabla_1 h_{ij}$ (by our chosen extension) New!

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Step 2: Preserving convexity (continued)

We then establish a convexity estimate for $\tilde{h}_{ij}$. We can exclude boundary minimum points as follow. Suppose $\tilde{h}_{11}$ is the boundary minimum.

$$\begin{aligned}\nabla_1 \tilde{h}_{11} = \nabla_1 h_{11} =& 2h^S_{22} H + (h^S_{\nu\nu} - 3h^S_{22}) h_{11} + \nabla^S_\nu h^S_{22} \\ \geq& (h^S_{\nu\nu} + h^S_{22}) h_{11} + \nabla^S_\nu h^S_{22} \\ =& H^S h_{11} + \nabla^S_\nu h^S_{22} > 0\end{aligned}$$

when h_{11} is sufficiently large. Similar calculation shows $\nabla_1 \tilde{h}_{22} > 0$. Thus any minimum of $\tilde{h}_{ij}$ is in the interior. Using the estimate

$$|(\partial_t - \Delta) P^\Sigma| \leq C_S (1 + |A|^2)$$

we obtain the convexity estimate via the maximum principle for tensors.

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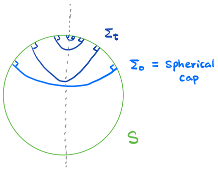
Step 3: Preserving pinching

Claim: $h_{ij} \geq \epsilon H g_{ij}$ is preserved under the flow for some $\epsilon > 0$

- **Huisken (1984)** proved this for any $\epsilon \in (0, 1/2]$ with $\epsilon = 1/2$ corresponding to the situation that Σ is totally umbilic (think about shrinking sphere). It is impossible to establish this optimal pinching estimate in the free boundary setting (even when $S = \mathbb{S}^2$):



- This loss of umbilicity phenomenon is due to the (in)-compatibility of the initial data at the boundary (the flow is only $C^{2+\alpha,1+\alpha/2}$ there). Note that

$$\frac{\partial}{\partial \eta} H = h^S_{\nu\nu} H$$

only holds for $t > 0$ unless there are higher order compatibility at $t = 0$.

Step 4: Stampacchia iteration

Finally, we use Stampacchia iteration to prove

Claim: $\frac{|A|^2 - \frac{1}{2}H^2}{H^{2-\sigma}}$ is uniformly bounded for all time for some $\sigma > 0$

- We need again to consider the perturbed $\tilde{A}$ and $\tilde{H}$.
- Using the claim, we can establish the following: for any $\eta > 0$,

$$|A|^2 - \frac{1}{2}H^2 \leq \eta H^2 + C(S, \eta, \Sigma_0)$$

$$|\nabla H|^2 \leq \eta H^4 + C(S, \eta, \Sigma_0)$$

from which our main result follows.

Thank you for your attention.

Credits: Picture credits to Colding - Minicozzi and Ben Andrews.

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