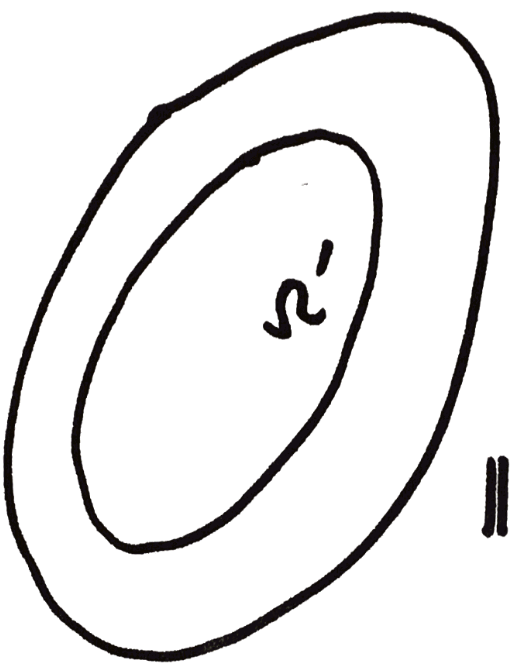
Asia-Pacific Analysis and PDE Seminar

Gradient estimates for the insulated conductivity problem

Nov 1, 9 pm, 2020, (EST)

Yanyan Li

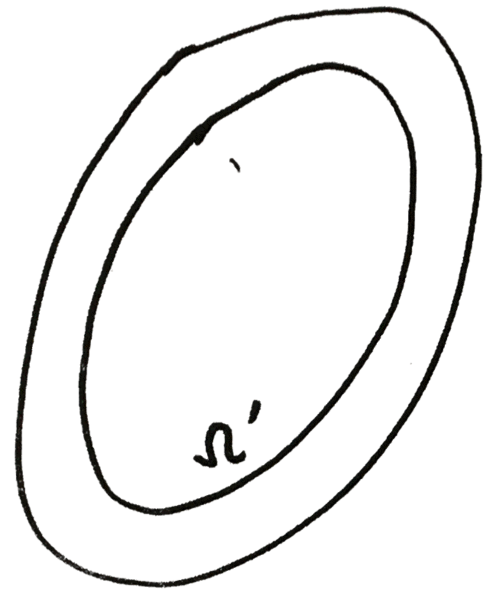


$\Omega \subset \mathbb{R}^n$

$\Delta u = 0, \ \Omega$

$\|u\|^2_{H^1(\Omega)} = \int_\Omega (|\nabla u|^2 + u^2) \leq 1$

$\Longrightarrow \quad \|\nabla u\|_{L^\infty(\Omega')} \leq C(\Omega, \Omega')$



$\Omega \subset \mathbb{R}^n$

$0 < \lambda \leq \Lambda < \infty$

elliptic

$\lambda I \leq (a_{ij}(x)) \leq \Lambda I$

$|\nabla a_{ij}(x)| \leq \Lambda$

$\partial_{x_i}\left(a_{ij}(x)\partial_{x_j} u\right) = 0, \quad \Omega$

$\|u\|_{H^1(\Omega)} \leq 1$

$\xRightarrow[\text{Schauder}]{} \quad \|\nabla u\|_{L^\infty(\Omega')} \leq C$

depends on

$\Omega \subset \mathbb{R}^n$

$0 < \lambda \leq \Lambda < \infty$

$\lambda I \leq (a_{ij}(x)) \leq \Lambda I$

$$\partial_{x_i}\left(a_{ij}(x)\,\partial_{x_j} u\right) = 0, \ \Omega$$

$$\|u\|_{H^1(\Omega)} \leq 1$$

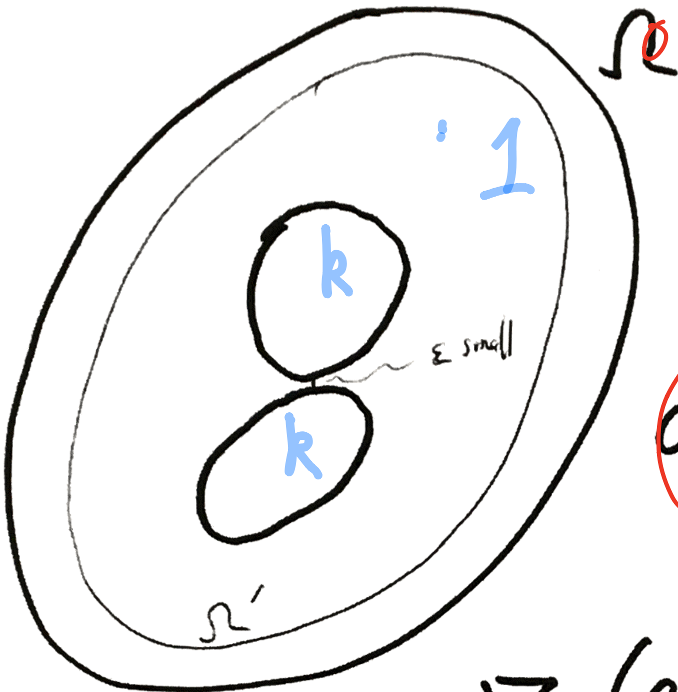
$\Longrightarrow$

$$\|u\|_{C^\alpha(\Omega')} \leq C$$

De Giorgi – Nash

$0 < \underline{\underline{\alpha}} < 1$

depends on $\underline{\lambda, \Lambda, \Omega, \Omega'}$



$0<\lambda\le a_k(x)\le\Lambda<\infty$

in $(0,\infty)$

$$a_k(x)=\begin{cases} k & D_1\cup D_2 \\ 1 & \Omega\setminus(D_1\cup D_2)\end{cases}$$

$$\nabla\cdot(a_k(x)\nabla u)=0,\ \Omega$$

$$\|u\|_{H^1(\Omega)}\le 1$$

$$\overset{?}{\Longrightarrow}\quad \|\nabla u\|_{L^\infty(\Omega')}\le C$$

independent of ε

- Does not follow from De Giorgi-Nash

transmission problem $\longrightarrow$ usual transmission problem ε-dependent

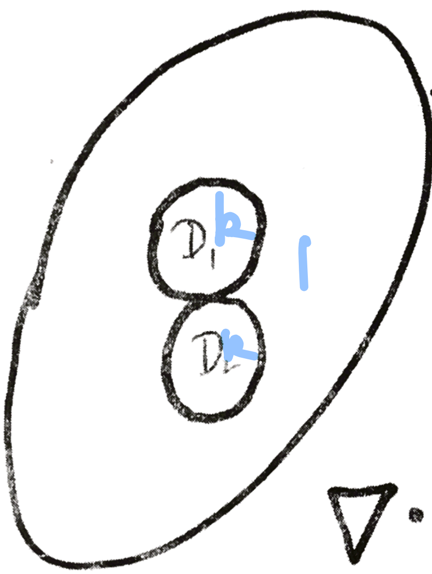
Babuska et al (1999)

(fiber reinforced material)

Observed numerically, for certain linear system of elasticity (Lame system),

$$\|\nabla u\|_{L^\infty} \leqslant C \text{ — independent of } \varepsilon$$

- Bonnetier and Vogelius (2000)



$\Omega \subset \mathbb{R}^2$, D_1 & D_2 balls,

$$a_k(x) = \begin{cases} k \in (0,\infty), & D_1 \cup D_2 \\ 1, & \Omega \setminus (D_1 \cup D_2) \end{cases}$$

$$\nabla \cdot \left(a_k(x) \nabla u\right) = 0, \ \Omega$$

$$u \in H^1(\Omega)$$

$$\Longrightarrow \quad \underline{\underline{\nabla u \in L^\infty(\Omega')}}, \ \Omega' \subset\subset \Omega$$

- Vogelius, — (2000)

  second order divergence form elliptic equations in all dimensions with piecewise smooth coefficients.

- Nirenberg, — (2003)

  second order divergence form elliptic systems, with piecewise smooth coefficients.
  (including linear systems of elasticity)

- Linear systems of elasticity:

$$\partial_\alpha \left(A_{ij}^{\alpha\beta}(x)\, \partial_\beta u^j \right) = 0$$

$$A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha} = A_{\alpha j}^{i\beta}$$

$x = (x^1, \dots, x^n)$

$u = (u^1, \dots, u^n)$

$1 \leq i, j, \alpha, \beta \leq n.$

$\partial_\alpha \equiv \partial_{x_\alpha}$

For all $n \times n$ symmetric matrices $\{\xi_\alpha^i\}$,

$$\lambda |\xi|^2 \leq A_{ij}^{\alpha\beta}(x)\, \xi_\alpha^i \xi_\beta^j \leq \Lambda |\xi|^2$$

$$0 < \lambda \leq \Lambda < \infty$$

- The Lame System:

$$\mu \Delta u + (\lambda + \mu) D(\operatorname{div} u) = 0$$

$\mu > 0$

$n\lambda + 2\mu > 0$

$x = (x_1, \cdots, x_n)$

$u = (u_1, \cdots, u_n)$

$\Omega \subset \mathbb{R}^n$

$u = g$ on $\partial\Omega$

smooth

$$\nabla \cdot (a_k(x) \nabla u) = 0, \quad \Omega$$

$$a_k(x) = \begin{cases} k \in [0, \infty), & D_1 \cup D_2, \\ 1, & \Omega \setminus (D_1 \cup D_2) \end{cases}$$

$\Longrightarrow$ • $\|u\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\partial\Omega)}$

(maximum principle)

• $\|\nabla u\|_{L^\infty(\Omega)} \leq C(k) \|g\|_{L^\infty(\partial\Omega)}$

(Vogelius, –) independent of ε

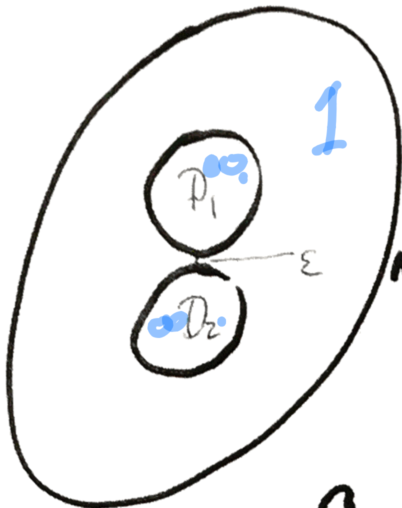
What happens if $k \to \infty$? (stiff fiber)

$k = \infty$ (perfect conductor)

$k \to 0$?

$k = 0$ (insulated conductor)

- $k = \infty$ (stiff fiber, perfect conductor)



$\Omega \subset \mathbb{R}^n$

D_1, D_2 strictly convex

$u = g$ on $\partial\Omega$

$$a_\infty(x) = \begin{cases} \infty, & D_1 \cup D_2 \\ 1, & \Omega \setminus (D_1 \cup D_2) \end{cases}$$

a_k, $u_k \to u$

$$\nabla \cdot \left(a_\infty(x) \nabla u \right) = 0, \quad \Omega.$$

- Budiansky & Carrier (1984)
  Markenscoff (1996)

$$|\nabla u|_{L^\infty} \sim \varepsilon^{-\frac{1}{2}} \quad \text{in dim } n = 2$$

special solutions

- Ammari, Kang, Lim (2005)
  Ammari, Kang, Lee, Lee, Lim (2007)
  Yun (2007, 2009)
  Ellen Bao, Biao Yin — (2009)

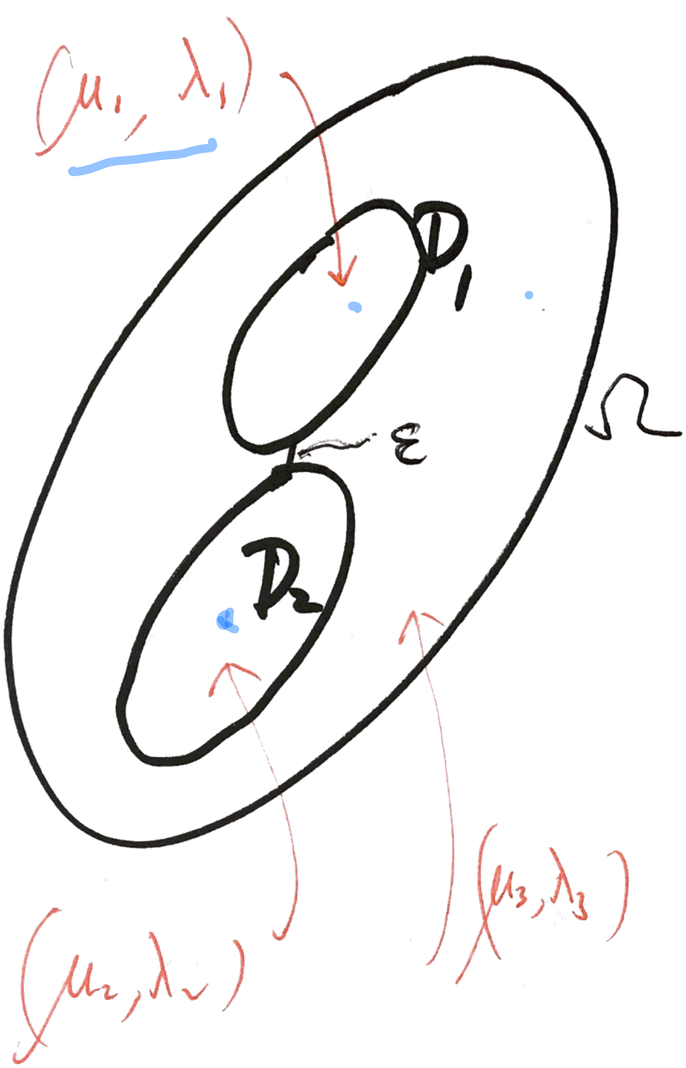
$$|\nabla u|_{L^\infty} \sim \begin{cases} \varepsilon^{-\frac{1}{2}}, & n = 2 \\ |\varepsilon \ln \varepsilon|^{-1}, & n = 3 \\ \varepsilon^{-1}, & n \geq 4 \end{cases}$$

Linear elasticity, the Lame system:

$$\mu \Delta u + (\lambda+\mu)\nabla(\mathrm{div}\, u) = 0,$$

$\mu > 0$

$n\lambda + 2\mu > 0$



$$\|\nabla u\|_{L^\infty} \leq C(\mu_i, \lambda_i)$$

$\mu_i > 0$

$n\lambda_i + 2\mu_i > 0$

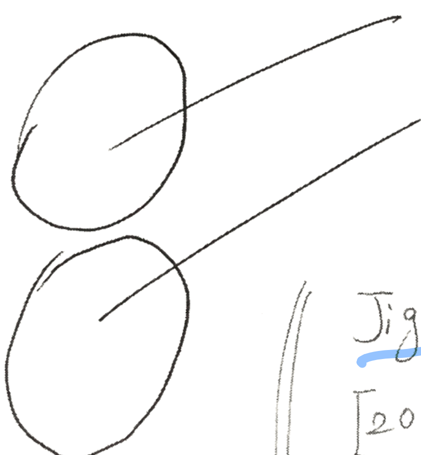
Nirenberg, — (2003)

For the Lame system with partially infinite coefficient:

$$|\nabla u|_{L^\infty} \le \begin{cases} \varepsilon^{-\frac{1}{2}}, & n=2 \\ |\varepsilon \ln \varepsilon|^{-1}, & n=3 \\ \varepsilon^{-1}, & n \geq 4 \end{cases}$$

$\mathcal{L}_{\lambda, \mu}$

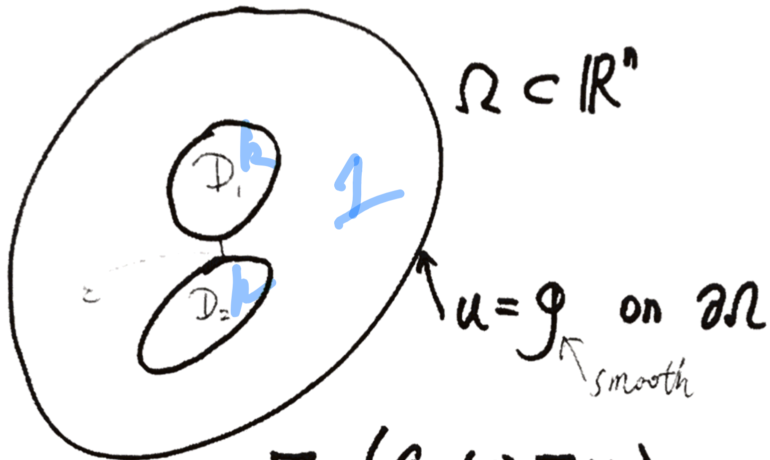
$$\left.\begin{array}{l} \mu > 0 \\ n\lambda + 2\mu > 0 \end{array}\right\} \text{finite}$$



$$\left.\begin{array}{l} \mu = \infty \\ n\lambda + 2\mu = \infty \end{array}\right\}$$

Jiguang Bao, Haigang Li, —

[2015, 2017]

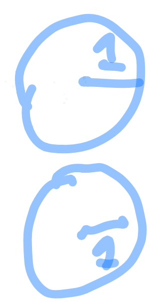


$$\nabla \cdot (a_k(x) \nabla u) = 0, \quad \Omega$$

$$a_k(x) = \begin{cases} k \in [0,\infty), & D_1 \cup D_2, \\ 1, & \Omega \setminus (D_1 \cup D_2) \end{cases}$$

For $n = 2$,

$\mathbb{R}^2$



- As $k \to \infty$,

$$|\nabla u| \leqslant \frac{C}{\frac{1}{k} + \sqrt{\varepsilon}}$$

$|u(x) - H(x)| \to O(|x|^{-1})$ as $|x| \to \infty$

$\Delta H = 0$ $\mathbb{R}^2$

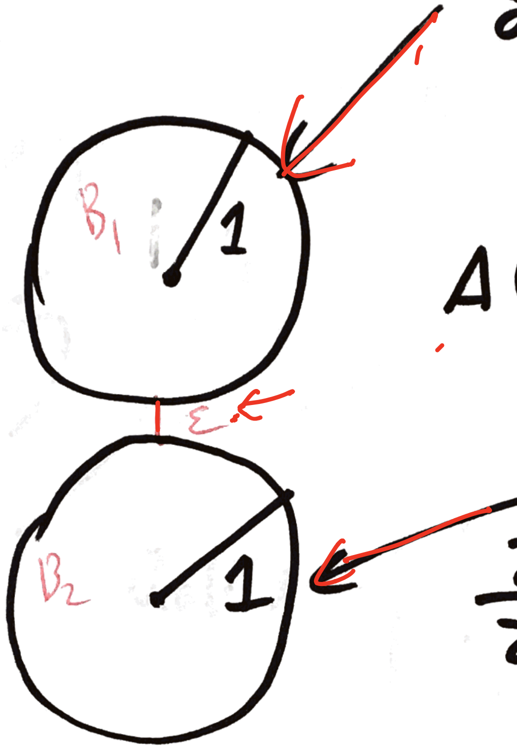
Ammari, Kang, Lim (2005)
Ammari, Kang, Lee, Lee, Lim (2007)
Hongjie Dong, Haigang Li (2019)

For $n \geqslant 3$, Open.

• $k = 0$ (insulated case)

$n = 2$.

$$\frac{\partial u}{\partial \nu} = 0$$



$$\Delta u = 0 \quad \mathbb{R}^2 \setminus \overline{(B_1 \cup B_2)}$$

$$\frac{\partial u}{\partial \nu} = 0$$

$\Delta H = 0$ in $\mathbb{R}^2$,

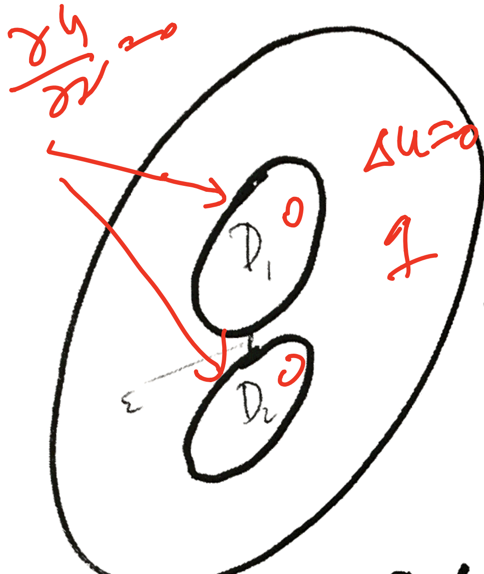
$$u(x) - H(x) = O(|x|^{-1})$$

as $|x| \to \infty$.

$$\Longrightarrow \quad |\nabla u| \leq \frac{C}{\sqrt{\varepsilon}}.$$

[(Ammari, Kang, Lim (2005))
(Ammari, Kang, Lee, Lee, Lim (2007))]

- $k=0$ (insulated case)



$\Omega \subset \mathbb{R}^n$, $n \geq 2$

D_1, D_2 strictly convex

$u=\varphi$ on $\partial\Omega$

$$a_0(x) = \begin{cases} 0, & D_1 \cup D_2 \\ 1, & \Omega \setminus (D_1 \cup D_2) \end{cases}$$

$$\nabla \cdot (a_0(x) \nabla u) = 0, \quad \Omega.$$

$$\Longrightarrow \quad \|\nabla u\|_{L^\infty} \leq \frac{C}{\sqrt{\varepsilon}}$$

Ellen Bao, Biao Yin, – (2010)

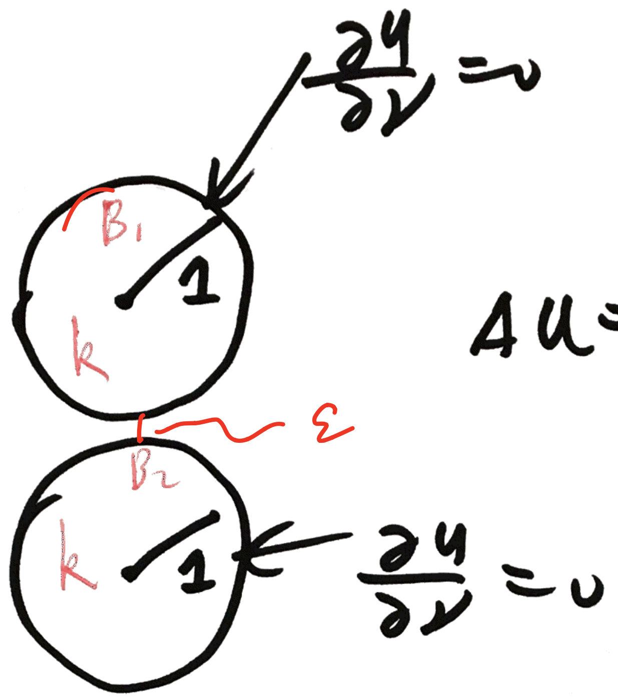
- $n=2$, sharp.

- Opened: $n \geq 3$, sharp?

for some time

$n = 2$

$k \to 0^+$



$\Delta u = 0 \quad \mathbb{R}^2 \setminus (\overline{B_1 \cup B_2})$

$\Delta H = 0$, $u(x) - H(x) = O(|x|^{-1})$

$\mathbb{R}^2$ as $|x| \to \infty$

$$\Longrightarrow \quad \boxed{|\nabla u| \le \frac{C}{k + \sqrt{\varepsilon}}}$$

[Ammari, Kang, Lim (2005)

Ammari, Kang, Lee, Lee, Lim (2007)

and also for two strictly convex domains D_1 and D_2

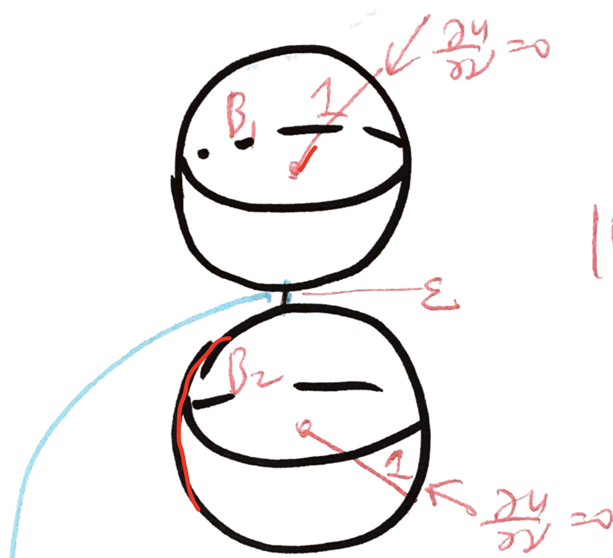
Localized by Hongjie Dong and Haigang Li (2019)

- $k=0$, (insulated)

$n \geq 3$, $|\nabla u| \leq \dfrac{C}{\sqrt{\varepsilon}}$ sharp ?

Bao, Yin, – (~~2016~~) $\approx$ 2009 $\approx$ 2010 ?

KiHyun Yun (2016) : $n = 3$

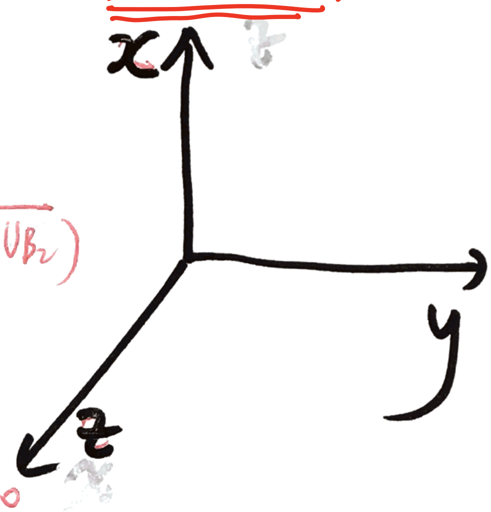


$(x, 0, 0)$

$\Delta u = 0$, $\mathbb{R}^3 \setminus (\overline{B_1 \cup B_2})$

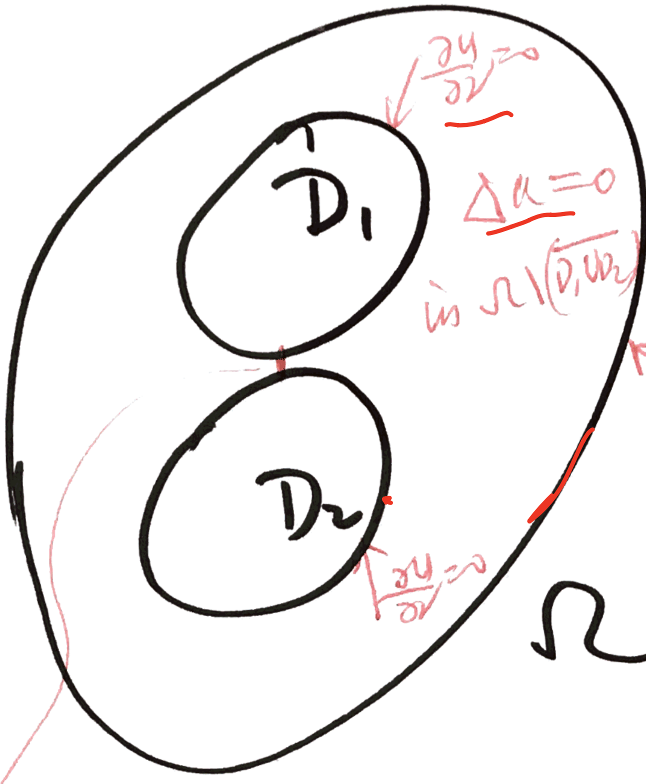
$|u(x,y,z) - y| \longrightarrow 0$

as $|(x,y,z)| \to +\infty$



$$|\nabla u(x,0,0)| \sim \frac{1}{\varepsilon^{\frac{2-\sqrt{2}}{2}}}$$

$$\frac{2-\sqrt{2}}{2} < \frac{1}{2}$$



$C^\infty(\partial\Omega)$

$u|_{\partial\Omega}=\varphi$

D_1, D_2 smooth, strictly convex,

$\Omega\subset\mathbb{R}^n$, $n\geq 3$

$\varepsilon>0$

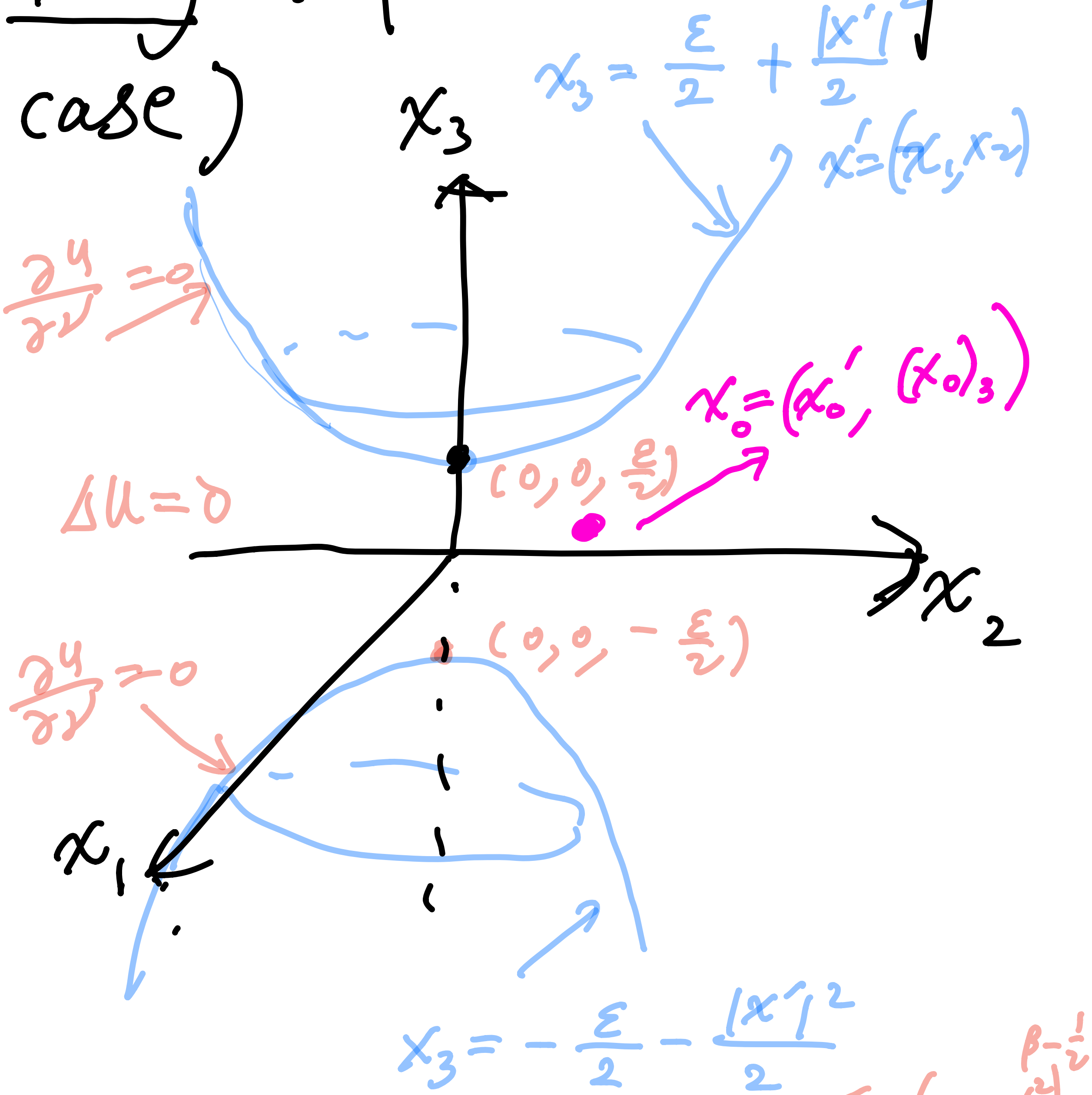
$\exists$ $\beta>0$, depending only on n, D_1, D_2, Ω.

$$\|\nabla u\|_{L^\infty(\Omega\setminus(\overline{D_1\cup D_2}))} \leq \frac{C\varepsilon^\beta}{\sqrt{\varepsilon}}\|\varphi\|_{L^\infty(\partial\Omega)}$$

Zhuolun Yang, —

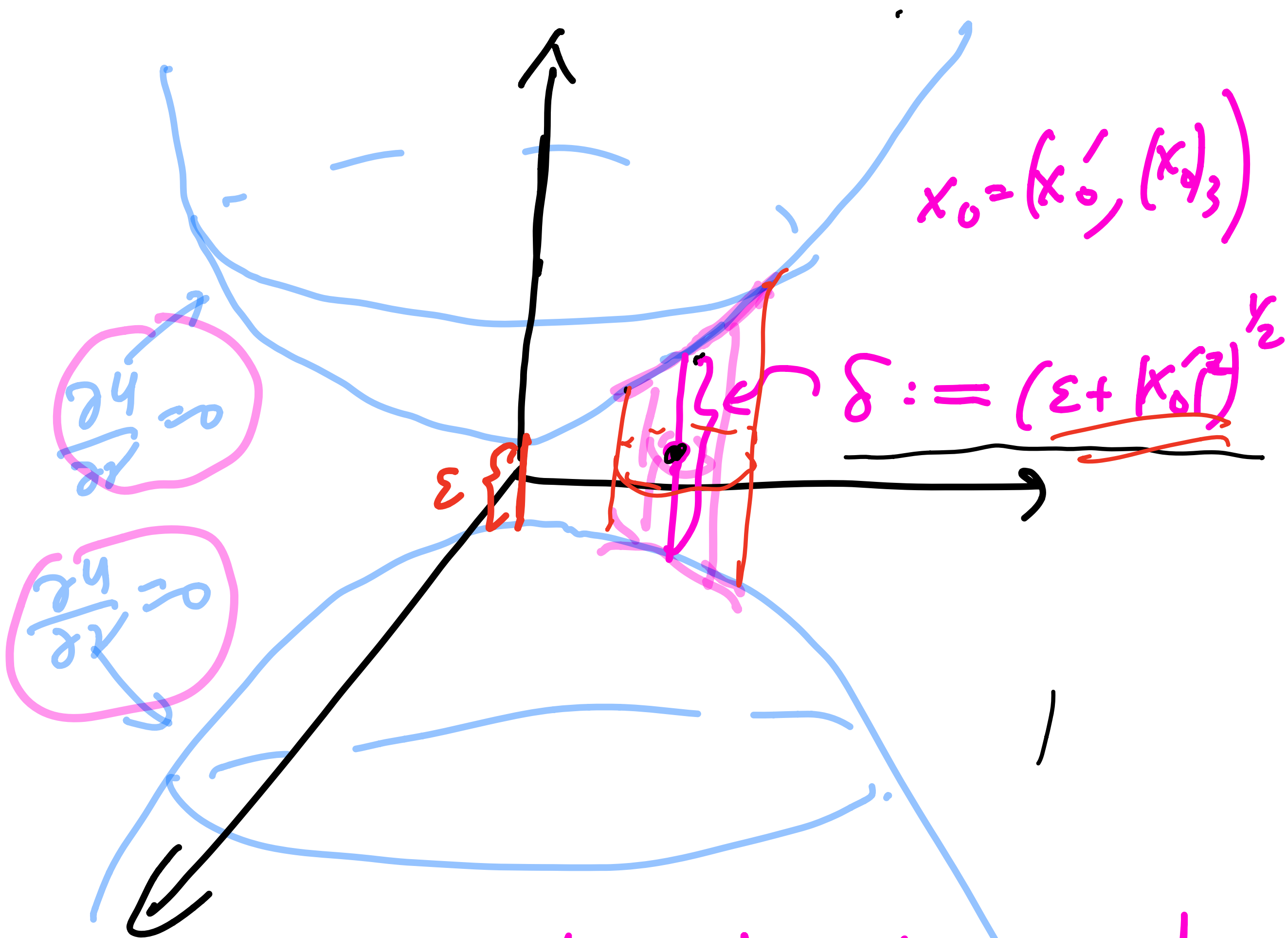
system $\|\nabla u\|_{L^\infty(\ )} \leq \frac{C}{\sqrt{\varepsilon}}\|u\|_{L^\infty(\ )}$

Proof: (Take a simpler case)



Will prove: $\exists\, \beta > 0$, $\sim C(\varepsilon + |x_0'|^2)^{\beta - \frac{1}{2}}$

$$|\nabla u(x_0)| \leq \frac{C \cdot \varepsilon^{\beta}}{\sqrt{\varepsilon}} \|u\|_{L^\infty}$$



$\gamma > 0$ a small number to be fixed, independent of ε.

$$\Omega_{x_0, r} := \left\{ -\left(\frac{\varepsilon}{2} + \frac{|x'|^2}{2}\right) < x_n < \left(\frac{\varepsilon}{2} + \frac{|x'|^2}{2}\right), \; |x' - x_0'| < r \right\}$$

Work with $\boxed{5|x_0'| < r < \delta^{1-\gamma}}$

w.l.g. $1 > u > 0$

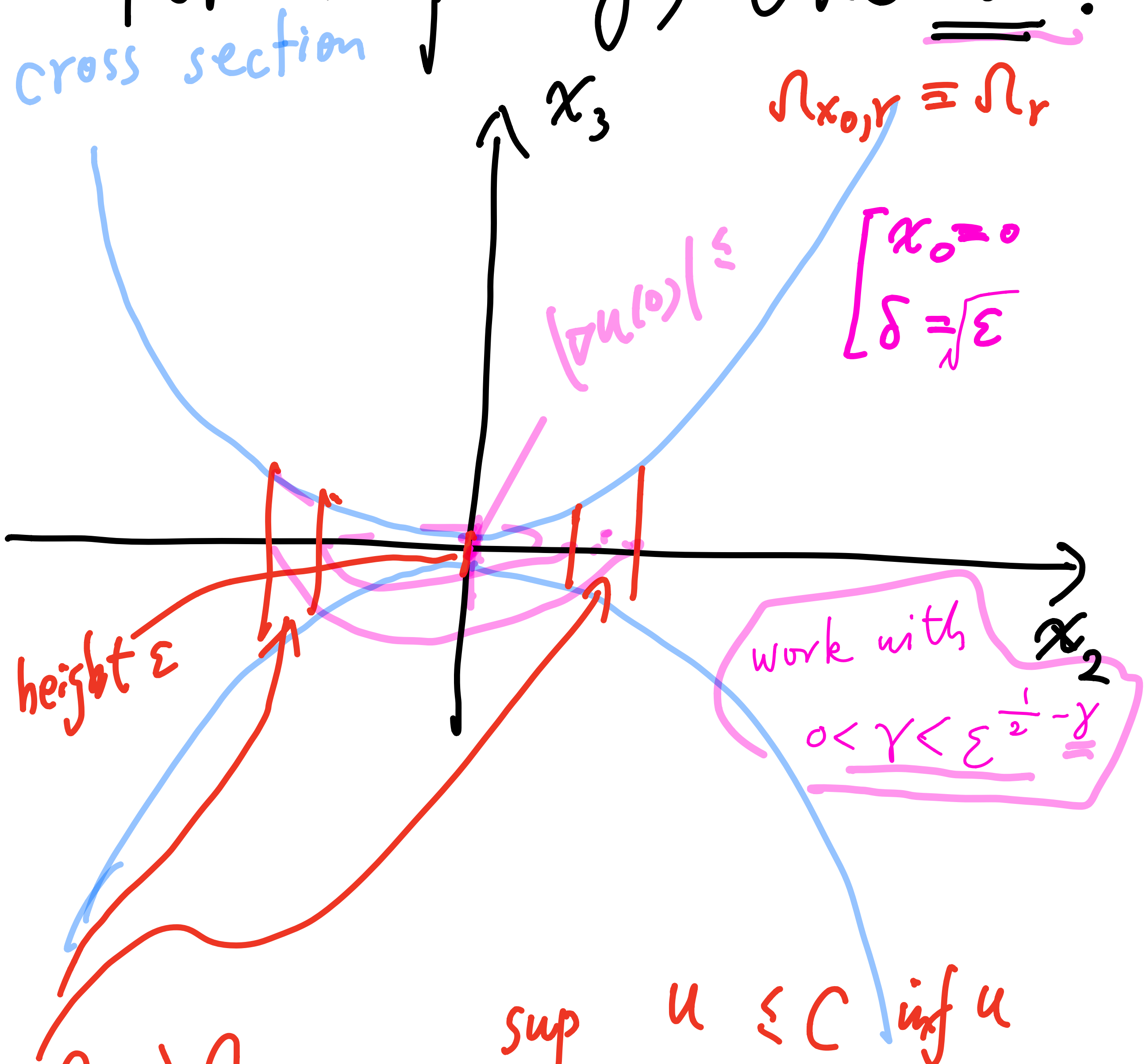
$$\begin{cases} \Delta u = 0 & \text{in } \Omega_{x_0, 2r} \setminus \Omega_{x_0, \frac{r}{4}}, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on top and bottom} \end{cases}$$

Harnack $\Longrightarrow$

$$\sup_{\Omega_{x_0,r} \setminus \Omega_{x_0,r/2}} u \le C \inf_{\Omega_{x_0,r} \setminus \Omega_{x_0,\frac{r}{2}}} u$$

independent of ε, r, u.

For simplicity, take $\underline{\underline{x_0 = 0}}$.



$$\sup_{\Omega_r \setminus \Omega_{r/2}} u \leq C \inf_{\Omega_r \setminus \Omega_{r/2}} u$$

maximum principle

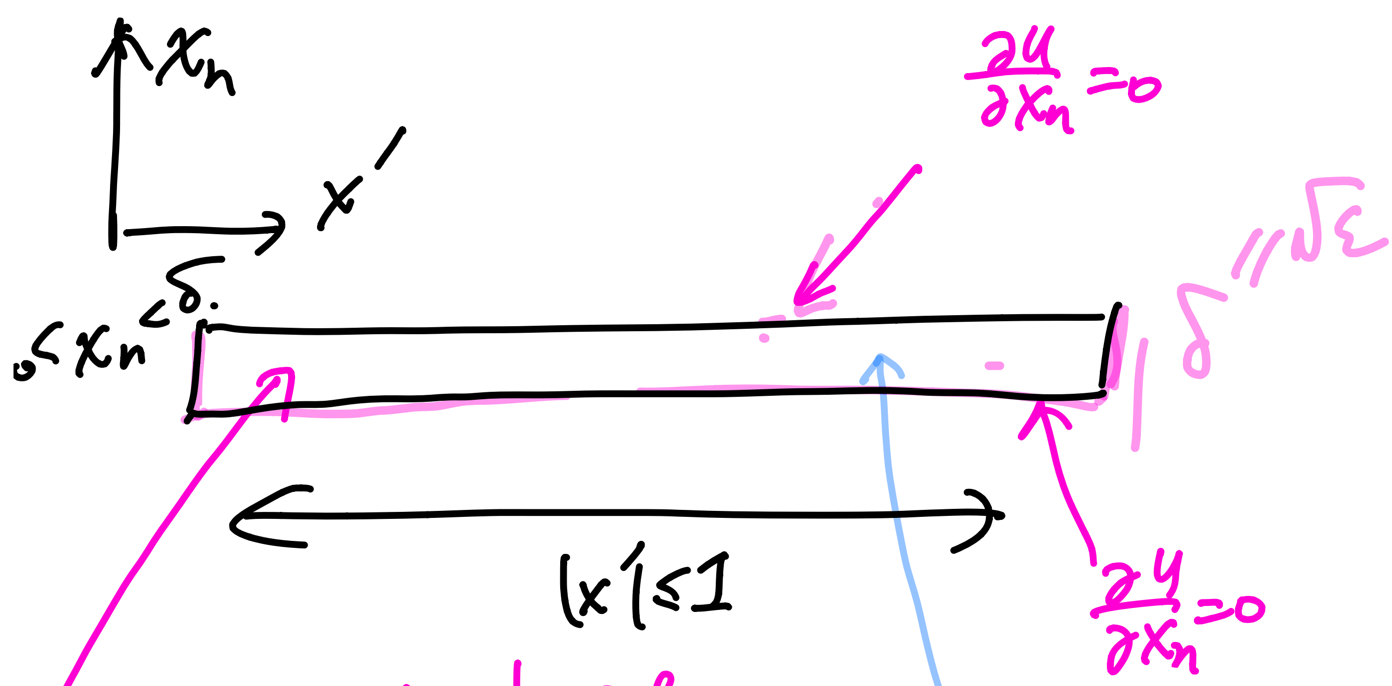
$$\Longrightarrow \sup_{\Omega_r} u \leq C \inf_{\Omega_r} u$$

$$\Longrightarrow \quad \operatorname{osc}_{\Omega_{\sqrt{\varepsilon}}} u \le C\varepsilon^{\beta'}$$

$\beta' > 0$

reduce to:

$\longrightarrow$:



$$\partial_i(a_{ij}\partial_j u) = h + \partial g$$

$$S = (-1,1)^{n-1}\times(0,\delta)$$

$$a_{nj}\partial_j u = 0$$

$$\|a_{ij}\|_{C^\alpha(S)} \le C_1, \quad 0<\alpha<1,$$

$$\|u\|_{L^\infty\left((-\frac{1}{2},\frac{1}{2})^{n-1}\times(0,\delta)\right)} + \|\nabla u\|_{L^\infty\left((-\frac{1}{2},\frac{1}{2})^{n-1}\times(0,\delta)\right)}$$

$$\le C\left(\frac{1}{\sqrt{\varepsilon}}\|u\|_{L^2(S)} + \|h\|_{L^\infty(S)} + \|g\|_{C^\alpha(S)}\right).$$

$\|u\|_{L^\infty(S)}$

such results were established in earlier joint work with Vogelius and Nirenberg.