



Instituto de
Matemáticas

Asymptotic analysis of fractional problems via the logarithmic Laplacian

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The fractional Laplacian, $s \in (0, 1)$

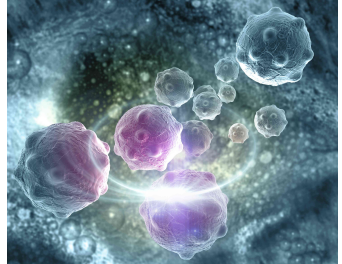
- Via Fourier transform

$$(-\Delta)^s u(x) = F^{-1}(|\cdot|^{2s} F(u))(x).$$

- Via a hypersingular integral

$$(-\Delta)^s u(x) = c_{N,s} p.v. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy.$$

Diffusion with jumps



The role of the parameter s to model diffusion strategies

(Caffarelli, Dipierro, Valdinoci; 2016)

$$(-\Delta)^s u = (\sigma - u)u + \tau(J \star u) \quad \text{in } \Omega, \quad u = 0 \quad \text{outside } \Omega.$$

- In **large** environments: $s \approx 1$ can be optimal.
- In **small** environments: $s \approx 0$ can be optimal.

The role of the parameter s to model diffusion strategies

(Pellacci, Verzini; 2016)

$$u_t + (-\Delta)^s u = m(x)u - u^2 \quad \text{in } \Omega, \quad \text{Neumann boundary conditions.}$$

If the habitat is

- not too **fragmented**, or
- not too **hostile**,

then the best choice for s is either the **smallest** or the **biggest** admissible values.

Some questions

How to study the s -dependency?

For example:

- What can be said about the map $s \mapsto (-\Delta)^s \varphi$?
- If u_s is a solution of

$$(-\Delta)^s u_s = f,$$

What can be said about the solution map $s \mapsto u_s$?, Is it continuous?, differentiable?, if so, how is the derivative $v_s := \partial_s u_s$ analyzed?

- How is this influenced by a nonlinear component?, for instance, if u_s is a solution of

$$(-\Delta)^s u_s = |u_s|^{p_s-2} u_s,$$

What can be said about the map $s \mapsto u_s$?, What happens as $s \rightarrow 0^+$?

Easy?

The fundamental solution for $(-\Delta)^s$ in \mathbb{R}^N is given by

$$F_{N,s}(x) := \begin{cases} \kappa_{N,s} |x|^{2s-N}, & \text{if } s - \frac{N}{2} \notin \mathbb{N}_0; \\ \kappa_{N,s} |x|^{2s-N} \ln |x|, & \text{if } s - \frac{N}{2} \in \mathbb{N}_0, \end{cases}$$

where

$$\kappa_{N,s} := \begin{cases} \frac{\Gamma(\frac{N}{2} - s)}{4^s \pi^{\frac{N}{2}} \Gamma(s)}, & \text{if } s - \frac{N}{2} \notin \mathbb{N}_0; \\ \frac{2^{1-2s} \pi^{-\frac{N}{2}} (-1)^{s+1-\frac{N}{2}}}{\Gamma(s - \frac{N}{2} + 1) \Gamma(s)}, & \text{if } s - \frac{N}{2} \in \mathbb{N}_0. \end{cases}$$

What can be said about
 $s \mapsto (-\Delta)^s \varphi$?

The logarithmic Laplacian: a pseudodifferential operator

Let $\varphi \in C_c^\infty(\mathbb{R}^N)$, $\xi \in \mathbb{R}^N \setminus \{0\}$, then

$$\begin{aligned}\widehat{(-\Delta)^s \varphi}(\xi) &= |\xi|^{2s} \widehat{\varphi}(\xi) = (1 + s \ln(|\xi|^2) + o(s)) \widehat{\varphi}(\xi) \\ &= \widehat{\varphi}(\xi) + s \ln(|\xi|^2) \widehat{\varphi}(\xi) + o(s)\end{aligned}$$

as $s \rightarrow 0^+$. Let

$$\widehat{\ln(-\Delta) \varphi}(\xi) = \ln(|\xi|^2) \widehat{\varphi}(\xi).$$

Other notations

$$\ln(-\Delta), \quad \log(-\Delta), \quad L_\Delta, \quad (-\Delta)^{\log}, \quad (-\Delta)^L.$$

An integrodifferential operator

$$L_{\Delta}\varphi(x) = \mathcal{F}^{-1}(\ln(|\cdot|^2)F(\varphi))(x).$$

(Chen, Weth; 2019)

$$L_{\Delta}\varphi(x) = c_N \int_{B_1(x)} \frac{\varphi(x) - \varphi(y)}{|x - y|^N} dy - c_N \int_{\mathbb{R}^N \setminus B_1(x)} \frac{\varphi(y)}{|x - y|^N} dy + \rho_N \varphi(x),$$

where c_N and ρ_N are explicit constants.

Derivative of $(-\Delta)^s$ at $s = 0$

$$L_{\Delta}\varphi(x) = c_N \int_{B_1(x)} \frac{\varphi(x) - \varphi(y)}{|x - y|^N} dy - c_N \int_{\mathbb{R}^N \setminus B_1(x)} \frac{\varphi(y)}{|x - y|^N} dy + \rho_N \varphi(x),$$

(Chen, Weth; 2019)

For $1 < p \leq \infty$ and $\varphi \in C_c^\infty(\mathbb{R}^N)$,

$$\lim_{s \rightarrow 0^+} \left\| \frac{(-\Delta)^s \varphi - \varphi}{s} - L_{\Delta} \varphi \right\|_p = 0.$$

Namely,

$$L_{\Delta} \varphi(x) = \left. \frac{d}{ds} \right|_{s=0} (-\Delta)^s \varphi(x).$$

Variational structure

$$L_{\Delta}\varphi(x) = c_N \int_{B_1(x)} \frac{\varphi(x) - \varphi(y)}{|x - y|^N} dy - c_N \int_{\mathbb{R}^N \setminus B_1(x)} \frac{\varphi(y)}{|x - y|^N} dy + \rho_N \varphi(x),$$

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let

$$\mathcal{E}(\varphi, \psi) := \frac{c_N}{2} \int_{\mathbb{R}^N} \int_{B_1(x)} \frac{(\varphi(x) - \varphi(y))(\psi(x) - \psi(y))}{|x - y|^N} dy dx.$$

Then,

$$\mathbb{H}(\Omega) := \{\varphi \in L^2(\Omega) : \mathcal{E}(\varphi, \varphi) < \infty, \varphi = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}$$

is a Hilbert space with scalar product $\mathcal{E}(\cdot, \cdot)$. Let

$$\mathcal{E}_L(u, v) := \mathcal{E}(\varphi, \psi) - c_N \int_{\mathbb{R}^N} \int_{\mathbb{R}^N \setminus B_1(x)} \frac{\varphi(x)\psi(y)}{|x - y|^N} dy dx + \rho_N \int_{\mathbb{R}^N} \varphi(y)\psi(y) dy.$$

Weak solutions

$$L_{\Delta} u = f \quad \iff \quad \mathcal{E}_L(u, \varphi) = \int_{\Omega} f \varphi \, dx \quad \forall \varphi \in C_c^{\infty}(\Omega),$$

$$L_{\Delta} u = \lambda u \quad \iff \quad \mathcal{E}_L(u, \varphi) = \lambda \int_{\Omega} u \varphi \, dx \quad \forall \varphi \in C_c^{\infty}(\Omega).$$

(Chen, Weth; 2019) There is a sequence of eigenvalues $\lambda_1^L < \lambda_2^L \leq \dots \leq \lambda_k^L \leq \dots$ and eigenfunctions $\xi_k^L \in \mathbb{H}(\Omega)$ such that

- $\{\xi_k^L\}$ is an orthonormal basis of $L^2(\Omega)$,
- $\xi_1^L > 0$,

Eigenfunctions

$$L_{\Delta}\xi_1^L = \lambda_1^L \xi_1^L$$

(Chen, Weth; 2019) If ξ_1^s is the first eigenfunction of $(-\Delta)^s$ and

$$(-\Delta)^s \xi_1^s = \lambda_1^s \xi_1^s,$$

then $\lim_{s \rightarrow 0^+} \lambda_1^s = 1$,

- $\lambda_1^L = \left. \frac{d}{ds} \right|_{s=0} \lambda_1^s$,
- $\xi_1^s \rightarrow \xi_1^L$ as $s \rightarrow 0^+$.

Maximum principle

$$L_{\Delta}\xi_1^L = \lambda_1^L \xi_1^L$$

(Chen, Weth; 2019)

- The operator L_{Δ} satisfies the maximum principle if and only if $\lambda_1^L > 0$.
- $\lambda_1^L \leq \ln(\lambda_1)$, where $\lambda_1 = \lambda_1(\Omega) > 0$ is the first eigenvalue of the Laplacian.
- $\lambda_1^L \leq \ln(\lambda_1) < 0$ if $\lambda_1 < 1$ (for Ω big, for instance).

Regularity

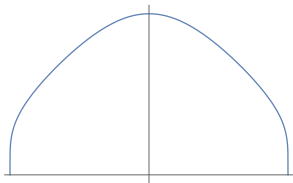
$$L_{\Delta} u = f$$

(Kassman, Mimica; 2013 - Chen, Weth; 2019)

- $f \in L^{\infty}(\Omega)$ implies that $u \in C(\bar{\Omega})$ and

$$|u(x)| \leq C_{\tau} \frac{1}{(-\ln \rho(x))^{\tau}} \quad \text{as } \rho(x) = \text{dist}(x, \partial\Omega) \rightarrow 0$$

for all $\tau \in (0, \frac{1}{2})$.



If u_s is a solution of

$$(-\Delta)^s u_s = f,$$

what can be said of $s \mapsto u_s$?

Differentiability of $s \mapsto u_s$

Let $z > 0$, $s \in [0, 1]$, and $h \in C^1(0, 1)$, then

$$\frac{d}{ds}(z^s h(s)) = \ln(z)z^s h(s) + z^s h'(s).$$

If

$$(-\Delta)^s u_s = f \quad \text{en } \Omega, \quad u_s = 0 \quad \text{on } \mathbb{R}^N \setminus \Omega,$$

then $s \mapsto u_s$ is differentiable in $(0, 1)$ (Jarohs, S., Weth; 2020) and

$$0 = \frac{d}{ds} \left((-\Delta)^s u_s \right) = \ln(-\Delta) \left((-\Delta)^s u_s \right) + (-\Delta)^s \left(\frac{d}{ds} u_s \right)$$

Then $v_s := \frac{d}{ds} u_s$ is the solution of

$$(-\Delta)^s v_s = -L_\Delta \left((-\Delta)^s u_s \right) \quad \text{in } \Omega, \quad v_s = 0 \quad \text{on } \partial\Omega.$$

Some results for $s \in [0, 1]$

$$v_s = \frac{d}{ds} u_s, \quad u_s = v_s = 0 \text{ in } \mathbb{R}^N \setminus \Omega,$$

$$(-\Delta)^s u_s = f \geq 0 \quad \text{in } \Omega, \quad (-\Delta)^s v_s = -L_\Delta \left((-\Delta)^s u_s \right) \quad \text{in } \Omega.$$

(Jarohs, S., Weth; 2020)

- $s \mapsto u_s(x)$ is decreasing in $[0, 1]$ for all $x \in \Omega \iff L_\Delta f \geq 0$ in Ω .
- if $f \equiv 1 \exists r_N > 0 : \Omega \subset B_{r_N}(0) \implies s \mapsto u_s$ is decreasing for $s \in [0, 1]$.
- if $f \equiv 1$ and $B_{r_N}(0) \subset \Omega$, then $s \mapsto u_s(0)$ is not decreasing in $[0, 1]$.

Note: $\frac{r_N}{\sqrt{N}} \rightarrow c > 0$ as $N \rightarrow \infty$.

Connecting the identity with the Laplacian

Let $\Omega \subset \mathbb{R}^2$ be an open bounded smooth set,

$$(-\Delta)^s u_s = 1 \geq 0 \quad \text{in } \Omega, \quad \frac{d}{ds} u_s = G_s \left(-L_\Delta \left((-\Delta)^s u_s \right) \right) \quad \text{in } \Omega,$$

where $G_s(h)(x) := \int_\Omega \mathcal{G}_s(x, y) h(y) dy$ and \mathcal{G}_s is the Green kernel for $(-\Delta)^s$ in Ω . Then, in Ω for $s \in (0, 1)$,

$$u_s - 1 = u_s - u_0 = \int_0^s \frac{d}{dt} u_t dt = \int_0^s G_t(-L_\Delta((-\Delta)^t u_t)) dt \leq -c_\Omega \int_0^s u_t dt,$$

where c_Ω is an explicit constant. Then,

$$u_s \leq 1 - c_\Omega \int_0^s u_t dt \quad \implies \quad u_s \leq e^{-s c_\Omega} \leq \frac{e^{-\frac{s}{5}}}{\pi^2} d_\Omega (\pi + |\Omega|),$$

for $s \in [0, 1]$, where

$$d_\Omega := \sup_{x \in \Omega} |B_1(x) \cap \Omega|$$

New bounds for the torsion

$$u_1(x) \leq d_\Omega(\pi + |\Omega|)\frac{9}{100}, \quad d_\Omega = \sup_{x \in \Omega} |B_1(x) \cap \Omega|.$$

Note that d_Ω is small if Ω is, for example, a [spaghetti-type domain](#).

This yields a better bound in these domains than the classical bound given by Talenti's principle

$$u_1(x) \leq \frac{|\Omega|}{4\pi} \leq \frac{8}{100}|\Omega|$$

New bound for the Laplacian's Green function

Let $\|G_1\|$ be the smallest constant $C > 0$ such that

$$|G_1 f|_\infty \leq C \quad \text{for all } f \in L^\infty(\Omega) \text{ such that } |f|_\infty := \sup_\Omega |f| \leq 1.$$

Since $|G_1 f|_\infty \leq |G_1 1|_\infty$ (by the maximum principle), we obtain that

$$\|G_1\| = |G_1 1|_\infty = \sup_\Omega u_1.$$

Consider the problem

$$-\Delta w = f(w) \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega, \quad (1)$$

where $f \in Lip(\mathbb{R})$ with Lipschitz constant $0 < C_f < \|G_1\|^{-1}$. Then,

$$T : L^\infty(\Omega) \rightarrow L^\infty(\Omega), \quad Tw := G_1 f(w),$$

is a **contraction**, since

$$|Tw_1 - Tw_2| \leq \|G_1\| |f(w_1) - f(w_2)|_\infty \leq \eta |w_1 - w_2|, \quad \eta = \|G_1\| C_f < 1,$$

and therefore (1) admits a unique solution in $L^\infty(\Omega)$.

If u_s is a solution of

$$(-\Delta)^s u_s = |u_s|^{p_s-2} u_s,$$

What can be said about $s \mapsto u_s$?,

What happens when $s \rightarrow 0^+$?

A superlinear subcritical problem

Let $p \in C^1([0, \frac{1}{4}])$ be such that

$$2 < p(s) < 2_s^* := \frac{2N}{N-2s} \quad \text{para } s \in (0, \frac{1}{4}), \quad p'(0) \notin \{0, \frac{4}{N}\}.$$

Let $p_s := p(s)$ and let u_s be the solution of

$$(-\Delta)^s u_s = |u_s|^{p_s-2} u_s \quad \text{in } \Omega, \quad u_s = 0 \text{ on } \mathbb{R}^N \setminus \Omega.$$

(S., Hernández Santamaría; 2022)

Let $s_k \rightarrow 0$ as $k \rightarrow \infty$ and let u_{s_k} be a **least-energy solution** of

$$(-\Delta)^{s_k} u_{s_k} = |u_{s_k}|^{p_{s_k}-2} u_{s_k} \quad \text{in } \Omega, \quad u_{s_k} = 0 \quad \text{on } \mathbb{R}^N \setminus \Omega.$$

Then there is $u_0 \in \mathbb{H}(\Omega) \setminus \{0\}$ such that (passing to a subsequence)

$$u_{s_k} \rightarrow u_0 \quad \text{en } L^2(\Omega)$$

and u_0 is a least-energy solution of

$$L_\Delta u_0 = p'(0) \ln(|u_0|) u_0 \quad \text{in } \Omega, \quad u_0 = 0 \quad \text{on } \mathbb{R}^N \setminus \Omega.$$

Some difficulties

- Lack of compactness (no embedding such as $\mathcal{H}_0^s(\Omega) \hookrightarrow L^{p_s}(\Omega)$).
- It only holds that $\mathbb{H}(\Omega) \hookrightarrow L^2(\Omega)$ is compact, but this is not enough.
- An important tool: the **logarithmic Sobolev inequality**.

A proof of the logarithmic Sobolev inequality

Step 1: Consider the fractional Sobolev inequality

$$\|u\|_{2_s^*}^2 \leq \kappa_{N,s} \|u\|_s^2 \quad \text{for all } u \in C_c^\infty(\mathbb{R}^N),$$

where $s \in (0, \frac{N}{2})$, $2_s^* := \frac{2N}{N-2s}$,

$$\|u\|_{2_s^*}^2 = \left(\int_{\mathbb{R}^N} |u|^{2_s^*} dx \right)^{\frac{2}{2_s^*}},$$

$$\|u\|_s^2 = \int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{u}(\xi)|^2 d\xi,$$

$$\kappa_{N,s} = 2^{-2s} \pi^{-s} \frac{\Gamma(\frac{N-2s}{2})}{\Gamma(\frac{N+2s}{2})} \left(\frac{\Gamma(N)}{\Gamma(\frac{N}{2})} \right)^{\frac{2s}{N}}.$$

A proof of the logarithmic Sobolev inequality

Step 1: Consider the fractional Sobolev inequality

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where $s \in (0, \frac{N}{2})$, $2_s^* := \frac{2N}{N-2s} = 2 + s\frac{4}{N} + o(s)$,

$$\begin{aligned} |u|_{2_s^*}^2 &= \left(\int_{\mathbb{R}^N} |u|^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \\ &= \int_{\mathbb{R}^N} |u|^2 dx + s \frac{4}{N} \left(\int_{\mathbb{R}^N} |u|^2 \ln |u| dx - \int_{\mathbb{R}^N} |u|^2 dx \ln \left(\int_{\mathbb{R}^N} |u|^2 dx \right) \right) + o(s), \end{aligned}$$

$$\begin{aligned} \|u\|_s^2 &= \int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{u}(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^N} |u|^2 dx + s \int_{\mathbb{R}^N} \ln(|\xi|^2) |\widehat{u}(\xi)|^2 d\xi + o(s), \end{aligned}$$

$$\begin{aligned} \kappa_{N,s} &= 2^{-2s} \pi^{-s} \frac{\Gamma(\frac{N-2s}{2})}{\Gamma(\frac{N+2s}{2})} \left(\frac{\Gamma(N)}{\Gamma(\frac{N}{2})} \right)^{\frac{2s}{N}} \\ &= (1 + s a_N + o(s)) \end{aligned}$$

for some $a_N \in \mathbb{R}$ as $s \rightarrow 0^+$.

A proof of the logarithmic Sobolev inequality

Step 2: A first order expansion on s :

$$|u|_{2_s^*}^2 \leq \kappa_{N,s} \|u\|_s^2 \quad \text{for all } u \in C_c^\infty(\mathbb{R}^N),$$

becomes

$$\begin{aligned} & \int_{\mathbb{R}^N} |u|^2 dx + s \frac{4}{N} \left(\int_{\mathbb{R}^N} |u|^2 \ln |u| dx - \int_{\mathbb{R}^N} |u|^2 dx \ln \left(\int_{\mathbb{R}^N} |u|^2 dx \right) \right) + o(s) \\ & \leq (1 + s a_N + o(s)) \left(\int_{\mathbb{R}^N} |u|^2 dx + s \int_{\mathbb{R}^N} \ln(|\xi|^2) |\widehat{u}(\xi)|^2 d\xi + o(s) \right) \\ & = \int_{\mathbb{R}^N} |u|^2 dx + s \left(a_N \int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^N} \ln(|\xi|^2) |\widehat{u}(\xi)|^2 d\xi \right) + o(s) \end{aligned}$$

A proof of the logarithmic Sobolev inequality

Step 3: Simplify:

$$\frac{4}{N} \left(\int_{\mathbb{R}^N} |u|^2 \ln |u| dx - |u|_2^2 \ln |u|_2^2 \right) \leq a_N |u|_2^2 + \int_{\mathbb{R}^N} \ln(|\xi|^2) |\widehat{u}(\xi)|^2 d\xi + o(1),$$

where

$$|u|_2^2 = \int_{\mathbb{R}^N} |u|^2 dx.$$

A proof of the logarithmic Sobolev inequality

Step 4: Let $s \rightarrow 0^+$:

$$\frac{2}{N} \int_{\mathbb{R}^N} |u|^2 \ln(|u|^2) dx \leq \int_{\mathbb{R}^N} \ln(|\xi|^2) |\widehat{u}(\xi)|^2 d\xi + \frac{2}{N} |u|_2^2 \ln |u|_2^2 + a_N |u|_2^2,$$

or

$$\frac{2}{N} \int_{\mathbb{R}^N} |u|^2 \ln(|u|^2) dx \leq \mathcal{E}_L(u, u) + \frac{2}{N} |u|_2^2 \ln |u|_2^2 + a_N |u|_2^2,$$

the Sobolev logarithmic inequality with optimal constant (Beckner, 1995).

The Nehari manifold method

$$L_{\Delta} u = \mu \ln(|u|)u, \quad \mu \in (0, \frac{4}{N}).$$

The energy functional $J : \mathbb{H}(\Omega) \rightarrow \mathbb{R}$ is given by

$$J(u) := \frac{1}{2} \mathcal{E}_L(u, u) - \frac{\mu}{4} \int_{\Omega} u^2 (\ln(|u|^2) - 1).$$

The Nehari manifold is

$$\mathcal{N} = \left\{ u \in \mathbb{H}(\Omega) \setminus \{0\} : \mathcal{E}_L(u, u) = \mu \int_{\Omega} \ln(|u|)u^2 \right\}.$$

Given $\varphi \in C_c^{\infty}(\Omega) \setminus \{0\}$,

$$t_{\varphi} \varphi \in \mathcal{N}, \quad \text{where } t_{\varphi} := \exp \left(\frac{\mathcal{E}_L(\varphi, \varphi) - \mu \int_{\Omega} \ln |\varphi| \varphi^2}{\mu |\varphi|_2^2} \right)$$

$$L_{\Delta}u = \mu \ln(|u|)u \quad \Longleftrightarrow \quad J(u) = \inf_{\mathcal{N}} J.$$

If $u \in \mathcal{N}$, then

$$J(u) := \frac{1}{2} \mathcal{E}_L(u, u) - \frac{\mu}{4} \int_{\Omega} u^2 (\ln(|u|^2) - 1) = \frac{\mu}{4} \int_{\Omega} u^2.$$

Proposition

Let $(u_n) \subset \mathcal{N}$ be such that $\sup_{n \in \mathbb{N}} J(u_n) \leq C$. Then (u_n) is bounded in $\mathbb{H}(\Omega)$.

$$J(u_n) = \frac{\mu}{4} |u_n|_2^2 \implies \sup_{n \in \mathbb{N}} |u_n|_2^2 \leq \frac{4}{\mu} C =: C_1.$$

By the Sobolev logarithmic inequality,

$$\begin{aligned} 2C > 2J(u_n) &= \mathcal{E}_L(u, u) - \frac{\mu}{2} \int_{\Omega} u^2 \ln(|u|^2) + \frac{\mu}{2} \int_{\Omega} u^2 \\ &\geq \left(1 - \frac{N\mu}{4}\right) \mathcal{E}_L(u_n, u_n) - \frac{\mu}{2} \ln(|u_n|_2^2) |u_n|_2^2 + \left(\frac{\mu}{2} - a_N \frac{N\mu}{4}\right) |u_n|_2^2. \end{aligned}$$

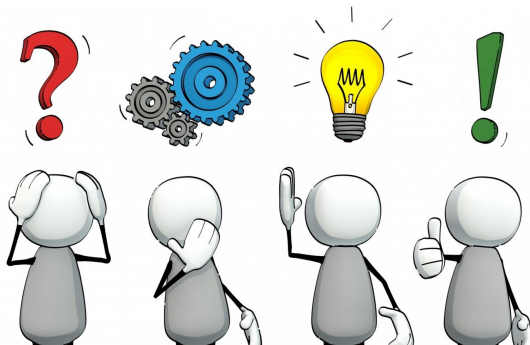
Then,

$$\sup_{n \in \mathbb{N}} \mathcal{E}_L(u_n, u_n) \leq \tilde{C} \left(1 + \sup_{t \in [0, C_1]} (|\ln(t)| + 1)t\right) =: C_2.$$

Thus,

$$\begin{aligned} C_2 &\geq \mathcal{E}_L(u_n, u_n) = \|u_n\|_{\mathbb{H}(\Omega)}^2 - c_N \int_{\mathbb{R}^N} \int_{\mathbb{R}^N \setminus B_1(x)} \frac{u_n(x)u_n(y)}{|x-y|^N} dy dx + \rho_N |u_n|_2^2 \\ &\geq \|u_n\|_{\mathbb{H}(\Omega)}^2 - (|\Omega| + |\rho_N|)C_1. \end{aligned}$$

Work in progress



Existence of classical solutions

(with Héctor Chang-Lara - CIMAT)

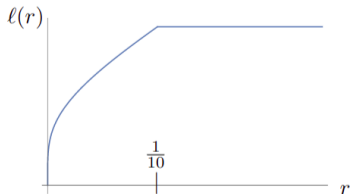
$$L_{\Delta}u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{outside } \Omega.$$

(Kassman, Mimica; 2013 - Chen, Weth; 2019)

$$f \in L^{\infty}(\Omega) \implies \sup_{x,y \in \Omega, x \neq y} \frac{|u(x) - u(y)|}{\ell^{\alpha}(|x - y|)} \quad \text{for some } \alpha \in (0, 1),$$

where

$$\ell(r) := \frac{1}{|\ln(\min\{r, \frac{1}{10}\})|}$$



$$L_{\Delta}u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{outside } \Omega.$$

(Jarohs, Feulefack; 2021)

$$f \in C^{2m}(\Omega) \implies u \in H_{loc}^m(\Omega),$$

in particular

$$f \in C^{\infty}(\Omega) \implies u \in C^{\infty}(\Omega),$$

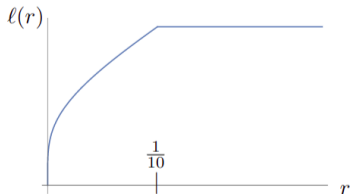
$$L_{\Delta}u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{outside } \Omega.$$

(Chang-Lara, S.; 2022)

$$\sup_{x,y \in \Omega, x \neq y} \frac{|f(x) - f(y)|}{\ell(|x - y|)} \implies \sup_{x,y \in \Omega, x \neq y} d^{1+\alpha}(x,y) \frac{|u(x) - u(y)|}{\ell^{1+\alpha}(|x - y|)}$$

for some $\alpha \in (0, 1)$, where

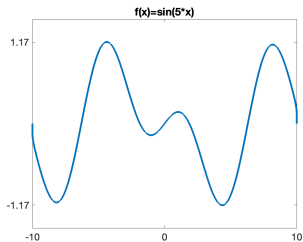
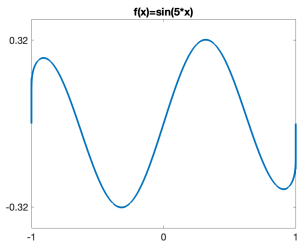
$$d(x, y) := \min\{\text{dist}(x, \partial\Omega), \text{dist}(y, \partial\Omega)\}, \quad \ell(\rho) := \frac{1}{\ln(\min\{\rho, \frac{1}{10}\})}$$



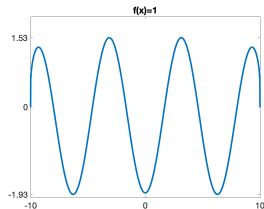
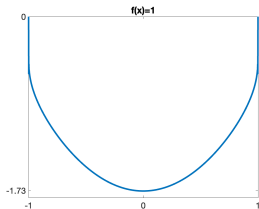
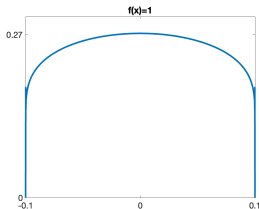
Numerical analysis of solutions

(with Víctor Hernández-Santamaría - IMUNAM)

$$L_{\Delta} u = \sin(5x) \quad \text{in } (-a, a), \quad u(-a) = u(a) = 0$$



$$L_{\Delta} u = 1 \quad \text{in } (-a, a), \quad u(-a) = u(a) = 0$$



Asymptotic analysis in sublinear-type problems

(with Felipe Angeles García - IMUNAM)

Some references (all available in arXiv)

- Chen, Huyuan, Tobias Weth. “**The Dirichlet problem for the logarithmic Laplacian.**” Communications in Partial Differential Equations 44.11 (2019): 1100-1139.
- Jarohs, Sven, A.S., Tobias Weth. “**A new look at the fractional Poisson problem via the logarithmic Laplacian.**” Journal of Functional Analysis 279.11 (2020): 108732.
- Laptev, Ari, Tobias Weth. “**Spectral properties of the logarithmic Laplacian.**” Analysis and Mathematical Physics 11.3 (2021): 1-24.
- Hernández-Santamaría, Víctor, A.S. “**Small order asymptotics for nonlinear fractional problems.**” To appear in Calculus of Variations and Partial Differential Equations. (2022)



Thank you!