Non-unique ergodicity for deterministic and stochastic 3D Navier–Stokes and Euler equations

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based on a joint work with R. Zhu and X. Zhu

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$$\begin{aligned} \mathrm{d}u + [\operatorname{div}(u \otimes u) + \nabla p]\mathrm{d}t &= \nu \Delta u \,\mathrm{d}t + \mathrm{d}B \\ \operatorname{div} u &= 0 \end{aligned} \qquad x \in \mathbb{T}^3, t \in \mathbb{R}$$

- trace class Brownian motion B on $(\Omega, \mathcal{F}, \mathbb{P})$
- velocity $u\colon \Omega \times \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}^3$
- pressure $p\colon \Omega \times \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$
- viscosity $\nu \geqslant 0$ – Navier–Stokes equations $\nu > 0$ and Euler equations $\nu = 0$
- high Reynolds number limit $\nu \to 0$ – highly turbulent regime

Statistically stationary solutions

- exact trajectories of solutions are not suitable for predictions (high sensitivity)
- statistical properties are well reproducible
- $\mathrm{Law}[u(t + \cdot)] = \mathrm{Law}[u(\cdot)]$ for all $t \in \mathbb{R}$ – as a pushforward probability measure on $C(\mathbb{R}; L^2)$

- physical theory taking theoretical hypotheses and making predictions
- confirmed to large extent by experiments
- largely open in terms of rigorous mathematics

Key problems of interest:

1. Existence and (non)uniqueness of ergodic stationary solutions u_ν to the Navier–Stokes equations
2. Relative compactness of stationary solutions u_ν, $\nu > 0$, and the convergence towards a stationary solution to the Euler equations
3. Anomalous dissipation along the vanishing viscosity limit $\nu \to 0$
4. Existence and (non)uniqueness of ergodic stationary solutions to the Euler equations

- up to now, results only for simplified settings
  - shell models of turbulence, passive scalar models of turbulence

- basic assumption in turbulence theory
- time averages along trajectories converge to ensemble averages wrt a probability measure

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T F(u(t))\mathrm{d}t = \int F\mathrm{d}\nu$$

- the measure is invariant – stationary solutions
- for an ergodic stationary solution

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T F(u(t))\mathrm{d}t = \mathbb{E}[F(u(0))]$$

- unique ergodicity for stoch. NSE with nondegenerate noise for a selected Markov process
  - Da Prato–Debussche '03 (analysis of the Kolmogorov equation)
  - Flandoli–Romito '08 (based on Markov selection)
- even mere existence of stationary solutions to stoch. Euler unknown

$$\begin{aligned} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p &= \nu \Delta u \\ \operatorname{div} u &= 0 \end{aligned} \qquad x \in \mathbb{T}^3, t \in [0, \infty)$$

- assume u is smooth – test the equation by u

$$\langle \partial_t u, u \rangle + \langle \operatorname{div}(u \otimes u), u \rangle + \langle \nabla p, u \rangle = \nu \langle \Delta u, u \rangle$$

$$\Rightarrow \qquad \frac{1}{2} \partial_t \|u\|_{L^2_x}^2 + \nu \|\nabla u\|_{L^2_x}^2 = 0$$

- energy conservation for Euler equations

$$\Rightarrow \qquad \frac{1}{2} \partial_t \|u\|_{L^2_x}^2 = 0$$

- vanishing viscosity limit in a class of smooth solutions would imply

$$\lim_{\nu \to 0} \nu \|\nabla u_\nu\|_{L^2_x}^2 = 0$$

- such solutions do not exist globally in time for general initial conditions
- Leray solutions to NSE exist globally in time and satisfy the energy inequality

$$\Rightarrow \qquad \frac{1}{2}\|u(t)\|_{L_x^2}^2 + \nu \int_0^t \|\nabla u(s)\|_{L_x^2}^2 \, \mathrm{d}s \leqslant \frac{1}{2}\|u(0)\|_{L_x^2}^2$$

- anomalous dissipation predicted by Kolmogorov

$$\lim_{\nu \to 0} \nu \mathbb{E}[\|\nabla u_\nu\|_{L_x^2}^2] = \epsilon > 0$$

- energy estimates do not give the necessary compactness to construct weak solutions to Euler

- we work with a different class of solutions (but not necessarily larger)
  - in $C(\mathbb{R}; H^\vartheta) \cap C^\vartheta(\mathbb{R}; L^2)$ for some (small) $\vartheta > 0$ uniformly in $\nu \geqslant 0$
- $u(t) \notin H^1$ and energy inequality not satisfied
- we use a new stochastic convex integration

- based on the convex integration by Buckmaster–Vicol '19
- iterative procedure, explicit construction of solutions scale by scale
- decomposition $u = z + v$

$$\mathrm{d}z - (\Delta - 1) z \mathrm{d}t = \mathrm{d}B, \qquad \mathrm{div}\, z = 0$$

$$\partial_t v - \Delta v - z + \mathrm{div}\,((v + z) \otimes (v + z)) + \nabla p = 0, \qquad \mathrm{div}\, v = 0$$

- iterations satisfy the equations up to an error

$$\partial_t v_q - \Delta v_q - z_q + \mathrm{div}\,((v_q + z_q) \otimes (v_q + z_q)) + \nabla p_q = \mathrm{div}\, R_q, \qquad \mathrm{div}\, v_q = 0$$

$$z_q = \mathbb{P}_{\leqslant f(q)} z$$

- having already found (v_q, R_q)
  - how to find (v_{q+1}, R_{q+1})?
  - so that also v_q has a limit and $R_q \to 0$?

- roughly speaking, we look for a (small) perturbation w_{q+1} so that
  - $v_{q+1} = v_q + w_{q+1}$
  - R_{q+1} is (much) smaller than R_q
- then looking at $\partial_t v_{q+1} - \partial_t v_q$ we get a formula for R_{q+1}

$$\operatorname{div} R_{q+1} = \operatorname{div}(R_q + w_{q+1} \otimes w_{q+1}) + \cdots$$

- intermittent jets W introduced by Buckmaster–Vicol, geometric lemma

$$w_{q+1} = a(R_q) W_{q+1}$$

- amplitude function $a(R_q)$ oscillates slowly, large oscillations in W_{q+1}
- large oscillations resonate through the nonlinearity so that

$$\|R_q + w_{q+1} \otimes w_{q+1}\| \ll \|R_q\|$$

- additionally: mollification step, compressibility and time corrector

- for the long time behavior, work with norms of the form

$$\left(\sup_{t\in\mathbb{R}}\mathbb{E}\left[\sup_{t\leqslant s\leqslant t+1}\|v_q(s)\|_{H^\vartheta}^{2r}\right]\right)^{1/(2r)}, \qquad \left(\sup_{t\in\mathbb{R}}\mathbb{E}[\|v_q\|_{C^\vartheta([t,t+1];L^2)}^{2r}]\right)^{1/(2r)}$$

- uniform moment estimates locally in $C(\mathbb{R};H^\vartheta)\cap C^\vartheta(\mathbb{R};L^2)$
- previous versions worked with stopping times – not good for stationary solutions
- iterative estimates (a sample): $r>1$ fixed, any $m\in\mathbb{N}$

$$\sup_{t\in\mathbb{R}}\mathbb{E}\left[\sup_{t\leqslant s\leqslant t+1}\|R_q(s)\|_{L^1}^{r}\right]\leqslant\frac{1}{48}\delta_{q+2}\to 0 \qquad \text{as} \qquad q\to\infty$$

$$\sup_{t\in\mathbb{R}}\mathbb{E}\left[\sup_{t\leqslant s\leqslant t+1}\|R_q(s)\|_{L^1}^{m}\right]\leqslant(6^q\cdot 4mL^2)^{(6^q)}$$

- due to the quadratic nonlinearity the estimates are superliner – control all the moments
- use small factors to absorb the blow up

H., Zhu, Zhu '22 *Let $r>1$ and a smooth $e\colon \mathbb{R}\to(0,\infty)$ with a compact range be given.*

There exists $\vartheta>0$ so that for every $\nu\geqslant 0$ there is an adapted $u_\nu\in C(\mathbb{R};H^\vartheta)\cap C^\vartheta(\mathbb{R};L^2)$ a.s. solving the stoch. NS/Euler equations so that

$$\sup_{\nu\geqslant 0}\left(\sup_{t\in\mathbb{R}}\mathbb{E}\left[\sup_{t\leqslant s\leqslant t+1}\|u_\nu(s)\|_{H^\vartheta}^{2r}\right]+\sup_{t\in\mathbb{R}}\mathbb{E}[\|u_\nu\|_{C^\vartheta([t,t+1];L^2)}^{2r}]\right)<\infty,$$

$$\mathbb{E}\|u_\nu(t)\|_{L^2}^2=e(t).$$

- solutions are probabilistically strong, analytically weak
- the bounds good enough to apply
  - Krylov–Bogoliubov – existence of stationary solutions
  - Krein–Milman – existence of ergodic stationary solutions
- nonuniqueness of the above by choosing different $e(t)=K$

- instead of Markov semigroup, work with shifts on trajectories

$$S_t(u,B)(\cdot) = (u(t+\cdot), B(t+\cdot) - B(t)) \qquad t \in \mathbb{R}$$

  - continuity on $\mathcal{T} = C(\mathbb{R}; L^2) \times C(\mathbb{R}; L^2)$ for free! (cf. Feller property)

- Krylov–Bogoliubov applied to the ergodic averages

$$\frac{1}{T}\int_0^T \mathcal{L}[S_t(u,B)]\mathrm{d}t \to \nu = \mathcal{L}[\tilde{u}, \tilde{B}] \qquad T \to \infty$$

- ν is a shift invariant measure on trajectories and a law of a stationary solution $(\tilde{u}, \tilde{B})$
- ergodicity understood as ergodicity of the dynamical system $(\mathcal{T}, (S_t, t \in \mathbb{R}), \mathcal{L}[\tilde{u}, \tilde{B}])$

- bounds uniform in the viscosity $\nu \geqslant 0$
  - the results apply to the stochastic Euler equations
  - vanishing viscosity limit in the framework of stationary solutions

Thanks for your attention!