Square functions and Riesz transforms on a class of non-doubling manifolds

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Abstract

Abstract: We consider a class of manifolds $\mathcal{M}$ obtained by taking the connected sum of a finite number of N-dimensional Riemannian manifolds of the form $(\mathbb{R}^{n_i}, \delta) \times (\mathcal{M}_i, g)$, where $\mathcal{M}_i$ is a compact manifold, with the product metric. The case of greatest interest is when the Euclidean dimensions n_i are not all equal. This means that the ends have different 'asymptotic dimension', and implies that the Riemannian manifold $\mathcal{M}$ is not a doubling space.We completely describe the range of exponents p for which the Riesz transform and vertical square function on $\mathcal{M}$ are bounded operators on $L^p(\mathcal{M})$.

Vertical square function

We demonstrate that the vertical square function operator

$$Sf(x) := \left(\int_0^\infty \left| t\nabla (I + t^2\Delta)^{-M} f(x) \right| 2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

is bounded on $L^p(\mathcal{M})$ for $1 < p < n_{min} = \min_i n_i$ and weak-type $(1, 1)$. In addition, it will be proved that the reverse inequality $||f||_p \lesssim ||Sf||_p$ holds for $p \in (n'_{min}, n_{min})$ and that S is unbounded for $p \geq n_{min}$ provided $2M < n_{min}$.

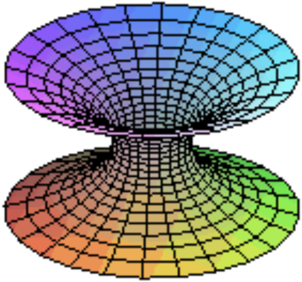
horizontal square function

Similarly, for $M > 1$, this method of proof will also be used to ascertain that the horizontal square function operator

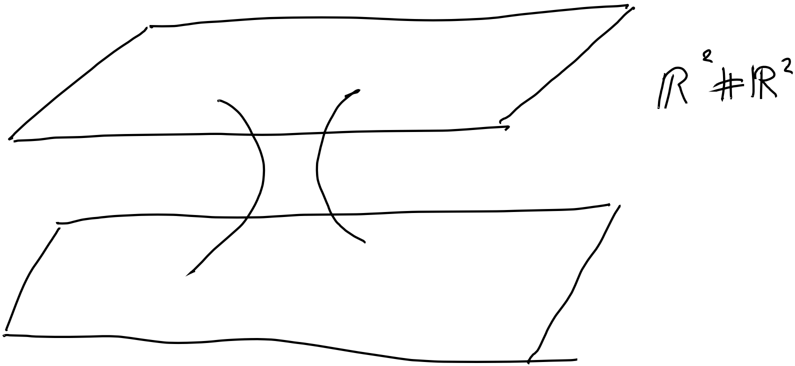
$$sf(x) := \left(\int_0^\infty \left| t^2 \Delta (I + t^2 \Delta)^{-M} f(x) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

is bounded on $L^p(\mathcal{M})$ for all $1 < p < \infty$ and weak-type $(1,1)$. Hence, for $p \geq n_{min}$, the vertical and horizontal square function operators are not equivalent and their corresponding Hardy spaces H^p do not coincide.

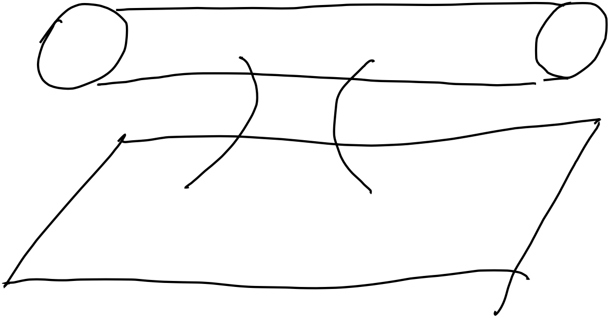
Manifolds with ends



Manifolds with ends



Manifolds with ends



$\mathbb{R}^1 \# \mathbb{R}^2$

If the dimensions at infinity for ends are not equal the doubling condition fails.

Riesz transform - Definitions

$\mathcal{M}$ complete Riemannian manifold and $d = \nabla$ the corresponding gradient.

Δ - Laplace-Beltrami operator defined by the following quadratic form

$$\langle \Delta f, g \rangle = \int_M g \Delta f d\mu = \int \nabla f \cdot \nabla g \, d\mu.$$

Riesz transform - Definitions

Riesz transform on is bounded on L^p:

$$(R_p) \qquad \||\nabla f|\|_p \le C\|\Delta^{1/2} f\|_p, \ \forall f \in L^p(X, \mu)$$

Note that $\||\nabla f|\|_2 = \|\Delta^{1/2} f\|_2$ so $(R_p 2)$ holds automatically.
Essentially we ask for the range of p such that $\nabla \Delta^{-1/2}$ is bounded on L^p

Doubling condition

Let us recall that a metric measured space $(\mathcal{X}, d, \mu)$ with metric d and Borel measure μ is said to satisfy the *doubling condition* that is if there exists universal constant C such that

$$(D) \qquad \mu(B(x, 2r)) \le C\mu(B(x, r)) \quad \forall\, r > 0,\ x \in X.$$

Here by $B(x, r)$ we denote the ball of radius r centred at $x \in X$.

In the setting we consider here the doubling condition fails
Also No Poincaré, No Harnack

Manifolds with ends

Grigor'yan and Saloff-Coste (re)introduced a notion of manifolds with ends and obtained satisfactory estimates for the corresponding heat kernels via probabilistic methods.

Our result I

Hassell-S.

Theorem

Suppose that $\mathcal{M} = (\mathbb{R}^{n_1} \times \mathcal{M}_1)\# \dots \#(\mathbb{R}^{n_l} \times \mathcal{M}_l)$ *is a manifold with* $l \geq 2$ *ends , with* $n_i \geq 3$ *for each i. Then the Riesz transform* $\nabla\Delta^{-1/2}$ *defined on* $\mathcal{M}$ *is bounded on* $L^p(\mathcal{M})$ *if and only if* $1 < p < \min\{n_1, \cdots, n_l\}$*. That is, there exists C such that*

$$\Big\| |\nabla\Delta^{-1/2} f| \Big\|_p \leq C\|f\|_p, \ \forall f \in L^p(X, \mu)$$

if and only if $1 < p < \min\{n_1, \cdots, n_l\}$*. In addition the Riesz transform* $\nabla\Delta^{-1/2}$ *is of weak type* $(1,1)$*.*

Key lemma

Lemma

Assume that each n_i *is at least* 3. *Let* $v \in C_c^\infty(\mathcal{M};\mathbb{R})$. *Then there is a function* $u\colon \mathcal{M}\times\mathbb{R}_+ \to \mathbb{R}$ *such that* $(\Delta + k^2)u = v$ *and such that, on the ith end we have:*

$$
\begin{aligned}
|u(z,k)| &\le C\langle d(z_i^\circ, z)\rangle^{-(n_i-2)} \exp(-kd(z_i^\circ, z)) \quad \forall z \in \mathbb{R}^{n_i} \times \mathcal{M}_i \\
|\nabla u(z,k)| &\le C\langle d(z_i^\circ, z)\rangle^{-(n_i-1)} \exp(-kd(z_i^\circ, z)) \quad \forall z \in \mathbb{R}^{n_i} \times \mathcal{M}_i
\end{aligned} \tag{1}
$$

for some $c, C > 0$.

Low Energy Resolvent Parametrix G_3 term

$$(\Delta + k^2)^{-1}(z,z') = G_1(k) + G_2(k) + G_1(k)$$

$$\begin{aligned}
G_1(k) &= \sum_{i=1}^{l} (\Delta_{\mathbb{R}^{n_i}\times\mathcal{M}_i} + k^2)^{-1}(z,z')\phi_i(z)\phi_i(z') \\
G_2(k) &= G_{int}(k)\Big(1 - \sum_{i=1}^{l}\phi_i(z)\phi_i(z')\Big) \qquad (2) \\
G_3(k) &= \sum_{i=1}^{l} (\Delta_{\mathbb{R}^{n_i}\times\mathcal{M}_i} + k^2)^{-1}(z_i^\circ, z')u_i(z,k)\phi_i(z').
\end{aligned}$$

G_1 is the resolvent acting on separate ends.
G_2 is the resolvent acting on the compact connecting part.
G_3 the interaction "perturbation" part.

Significance of the G_3 term

The range of p for which the Riesz transform is bounded on L^p is governed by the asymptotics of $G_3(k)$ part.
$G_3(k)$ is easy to understand because it (essentially)is of rank one that is

$$G_3(k)(z, z') = f_k(z)g_k(z)$$

for some function f_k and g_k.

Vertical Square function

For $M \in \mathbf{N}^*$, the vertical square function operator is defined:

$$Sf(x) := \left(\int_0^\infty \left| t\nabla (I + t^2\Delta)^{-M} f(x) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}. \tag{3}$$

The notion of square functions forms an essential part of harmonic analysis and has numerous applications, from the definition of Hardy spaces to providing an equivalent characterisation of the bounded holomorphic functional calculus of a sectorial operator.

Vertical Square function

Theorem

Let $\mathcal{M} = (\mathbb{R}^{n_1} \times \mathcal{M}_1)\# \cdots \#(\mathbb{R}^{n_l} \times \mathcal{M}_l)$ be a manifold with ends. For $M \in \mathbf{N}^*$, the vertical square function operator S, will satisfy the following properties:

(i) *S is bounded on $L^p(\mathcal{M})$ for all $p \in (1, n_{min})$, where $n_{min} := \min_i n_i$, and weak-type $(1,1)$;*

(ii) *If $2M < n_{min}$ then S is unbounded on $L^p(\mathcal{M})$ for $p \geq n_{min}$; and*

(iii) *For $p \in (n'_{min}, n_{min})$, there exists $c > 0$ for which*

$$\|f\|_p \leq c \, \|Sf\|_p$$

for all $f \in L^p(\mathcal{M})$.

Horizontal Square function

As an alternative to the vertical square function S, one can also consider the horizontal square function for $M > 1$:

$$sf(x) := \left(\int_0^\infty \left| t^2 \Delta (I + t^2 \Delta)^{-M} f(x) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}. \tag{4}$$

For this operator, we will prove the following theorem.

Horizontal Square function

Theorem

Let $\mathcal{M} = (\mathbb{R}^{n_1} \times \mathcal{M}_1)\# \cdots \#(\mathbb{R}^{n_l} \times \mathcal{M}_l)$ be a manifold with ends and fix $M > 1$. For any $p \in (1, \infty)$ the square function operator s, as defined in (4), is bounded on $L^p(\mathcal{M})$ and there exists $c,\ C > 0$ such that for all $f \in L^p(\mathcal{M})$

$$c \|f\|_p \leq \|sf\|_p \leq C \|f\|_p .$$

In addition, the operator s is of weak-type $(1, 1)$.

Open problem

L considered Laplace Beltrami operator and F nice compactly supported function F supp $F \subset (a, b)$, where $0 < a < b$. Then

$$\sup_{t>0} \|F(t\Delta)\|_{p\to p} \le C?$$

Interesting example (forget support)

$$\|\exp(-t(1+i)\Delta)\|_{p\to p} \le C?$$

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