Estimates and Geometry for a Free Surface Problem of Fluids with heat-conductivity

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Free Boundary Problem of Incompressible Euler Equations

$$\begin{cases} D_t v_j + \partial_j p = -g_j, \;\; j = 1, \cdots, n, \; \mathrm{div}\, v = 0, \; \text{in } \mathfrak{D}_t, \\ p = 0 \quad \text{on } \Gamma(t) := \partial \mathfrak{D}_t, \\ \mathscr{V}(\Gamma(t)) = v \cdot \mathscr{N}, \\ v = v_0 \quad \text{on } \mathfrak{D}_0. \end{cases} \tag{0.1}$$

for space dimension $n = 2, 3$, $v = (v_1, \cdots, v_n)$: fluid velocity, p: fluid pressure. $g_j = \delta_{jn} g$: gravity constant, $D_t := \partial_t + v^k \partial_k$. The Einstein summation convention is used.

$\mathfrak{D}_t \subset \mathbb{R}^n$: the changing domain occupied by the fluid at time t.

$\mathscr{V}(\Gamma(t))$: normal velocity of $\Gamma(t)$, $\mathscr{N}$: exterior unit normal vector to $\Gamma(t)$

- LWP or GWP (water wave) : irrotational, incompressible with gravity, no surface-tension, (infinite depth):

  – Wu (2d: Invent. Math. 1997, 2009; 3d: JAMS 1999, Invent. Math. 2011);

  – Germain, Masmoudi and Shatah (3d: Annal. Math.2012);

  – Ionescu and Pusateri (2d: Invent. Math.2014);

  – Lannes (JAMS 2005, M-d, finite depth).

  – M. Ming-C. Wang:

  Water waves problem with surface tension in a corner domain.

Very active recently on the global solution (irrotational) : with surface tension, finite depth.....:

Alazard-Delort (Ann. Sci. EC. Norm. Super(2015)), Alvarez-Samaniego- D. Lannes(Invent. Math.), Y. Deng, A.D. Ionescu, B. Pausader, F. Pusateri, (Acta Math. (2017)), P. Germain, N. Masmoudi, J. Shatah (CPAM 2015), A. Ionescu, F. Pusateri (CPAM 2016), A. Ionescu, F. Pusateri (Mem. Amer. Math. Soc, 2018), X. C Wang (Anal. PDE 10 (4) (2017), CPAM 2018, Adv. in Math. 2019).

Long-time solution for incompressible Euler-Poisson

Bieri, Lydia; Miao, Shuang; Shahshahani, Sohrab; Wu, Sijue, Comm. Math. Phys. 355 (2017).

- A priori estimates or LWP: without the assumption of irrotationality :
  – Christodoulou and Lindblad (CPAM 2000);
  – Lindblad (CPAM 2003, Annal. Math 2005);
  – Coutand and Shkoller (JAMS 2007);
  – Shatah and Zeng (CPAM 2008);
  – P. Zhang and Z. Zhang (CPAM2008);
- Finite time singularities
  Castro-Cordoba-Fefferman-Gancedo- Gomez-Serrano (Ann. of Math. (2013))

- Long-time solution with point-vortex
  Qingtang Su, arxiv.
- On tidal energy in Newtonian two-body motion
  Miao, Shuang; Shahshahani, Sohrab, Camb. J. Math. 7 (2019), no. 4, 469–585.
- Wang-Xin, Magnetic effects induced dissipation mechanism leading to global existence.

Taylor Sign Condition

$$\partial_{\mathcal{N}} p \leq -\varepsilon < 0 \text{ on } \partial\mathfrak{D}_t. \tag{0.2}$$

plays a crucial role in establishing the above well-posedness results without surface tension. For irrotational flow $\partial_i v^j = \partial_j v^i$ $(i, j = 1, \cdots, n)$ of incompressible Euler equations, the Taylor sign condition holds automatically if $\partial\mathfrak{D}_t$ is suitably smooth. For example, when $\mathfrak{D}_t$ is a bounded simply connected domain,

$$\Delta p = -\partial_i v^j \partial_j v^i, \text{ in } \mathfrak{D}_t, \ p = 0, \text{ on } \partial\mathfrak{D}_t.$$

In the irrotational case, $\Delta p < 0$ in $\mathfrak{D}_t$, the strong maximal principle implies that $\partial_{\mathcal{N}} p < 0$ on $\partial\mathfrak{D}_t$.

Related Results for Incompressible Navier-Stokes equations

Local and global well-posedness, long time behavior

T. Beal, Y. Guo-I. Tice, Solonnikov, Tani and ....

G. Gui (recent global result by using Lagrangian coordinates).

Vanishing viscosity limit

Masmoudi-Rosset (incompressible), Y. Wang- Z. Xin (both vanishing viscosity and zero surface tension) , Mei-Wang-Xin (compressible NS).

A free surface problem for highly subsonic heat-conductive inviscid flows -Joint with Huihui Zeng

The system

$$\begin{aligned} &(\partial_t + v^k\partial_k)v_j + \mathscr{T}\partial_j p = 0, \quad j = 1, \cdots, n \\ &\mathrm{div}\, v = \kappa\Delta\mathscr{T}, \quad (\partial_t + v^k\partial_k)\mathscr{T} = \kappa\mathscr{T}\Delta\mathscr{T} \end{aligned} \tag{0.3}$$

is used to approximate heat-conducting inviscid flows when the Mach number is small, i.e., highly subsonic (n: space dimension, $n = 2, 3$).

$v = (v_1, \cdots, v_n)$: velocity field

$\mathscr{T}$: temperature, p: pressure

$\kappa > 0$: heat-conductive coefficient (scaled), constant.

Free Surface Problem

Free surface problem for $n=2$ and $n=3$ for system (0.3):

$$\begin{aligned}
&(\partial_t + v^k\partial_k)v_j + \mathscr{T}\partial_j p = 0, \ \ j=1,\cdots,n \ \text{ in } \ \mathfrak{D}, \\
&\mathrm{div}\, v = \kappa\Delta\mathscr{T}, \ \ (\partial_t + v^k\partial_k)\mathscr{T} = \kappa\mathscr{T}\Delta\mathscr{T}, \ \text{ in } \ \mathfrak{D},
\end{aligned} \tag{0.4}$$

Unknowns to be determined: the velocity field $v=(v_1,\cdots,v_n)$, the temperature $\mathscr{T}$, the pressure p and the domain $\mathfrak{D}\subset[0,T]\times\mathbb{R}^n$.

Free Surface Problem

Given a simply connected bounded domain $\mathfrak{D}_0 \subset \mathbb{R}^n$ and initial data $(v_0, \mathscr{T}_0)$ satisfying $\operatorname{div} v_0 = \kappa \Delta \mathscr{T}_0$, we want to find a set $\mathfrak{D} \subset [0, T] \times \mathbb{R}^n$, a vector field v and a scalar function $\mathscr{T}$ solving (0.4) and satisfying

the initial conditions:

$$\mathfrak{D}_0 = \{x : (0, x) \in \mathfrak{D}\}, \ (v, \mathscr{T}) = (v_0, \mathscr{T}_0) \ \text{ on } \ \{0\} \times \mathfrak{D}_0, \quad (0.5)$$

boundary condition:

let $\mathfrak{D}_t = \{x \in \mathbb{R}^n : (t, x) \in \mathfrak{D}\}$, we require the boundary conditions on the free boundary $\partial \mathfrak{D}_t$,

$$p = 0, \quad \mathscr{T} = \mathscr{T}_b \ \text{ and } \ v_{\mathscr{N}} = \varpi \qquad \text{on } \ \partial \mathfrak{D}_t \qquad (0.6)$$

for each t. $\mathscr{N}$: the exterior unit normal to $\partial \mathfrak{D}_t$, $v_{\mathscr{N}} = \mathscr{N}^i v_i$, ϖ : the normal velocity of $\partial \mathfrak{D}_t$, $\mathscr{T}_b$: a positive constant.

Goal

Prove *a priori* bounds for problem (0.4)-(0.8) in Sobolev spaces for the initial data satisfying

$$\min_{\partial \mathfrak{D}_0} (-\partial_{\mathcal{N}} p) > 0, \tag{0.7}$$

which implies, as we will prove, for some $T > 0$ and $0 \leq t \leq T$,

$$-\partial_{\mathcal{N}} p \geq \varepsilon_b > 0 \text{ on } \partial \mathfrak{D}_t, \tag{0.8}$$

for $\varepsilon_b = \frac{1}{2} \min_{\partial \mathfrak{D}_0} (-\partial_{\mathcal{N}} p)$, where $\partial_{\mathcal{N}} = \mathcal{N}^j \partial_j$.

Remarks

1. Comparison with the incompressible Euler Equations and isentropic compressible Euler equations

a) Loss of symmetry of Equations, no any conservation laws;

In fact, for the incompressible Euler equations,

$$\frac{d}{dt}\int_{\mathfrak{D}_t}\frac{1}{2}|v|^2dx = 0,$$

and for the compressible isentropic Euler equations

$$\frac{d}{dt}\int_{\mathfrak{D}_t}\rho(\frac{1}{2}|v|^2 + e(\rho))dx = 0.$$

Those energy principles are the starting points for the problems, which are not only for the above two problems, but also for other related free boundary problems (e. g. MHD).

Remarks

However, for the problem we are studying, one only has

$$\frac{d}{dt}\int_{\mathfrak{D}_t} \frac{1}{2\mathscr{T}}|v|^2 dx = \int_{\mathfrak{D}_t} p \operatorname{div} v dx.$$

This attributes to the loss of one more derivative than the problems on incompressible and isentropic compressible Euler equations.

Remarks

b) More careful analysis of loss of one more derivative than the problems for incompressible Euler and isentropic compressible Euler equations.

For the incompressible Euler, the following Dirichlet problem

$$\Delta p = -\partial_i v^j \partial_j v^i, \ in\, \mathfrak{D}_t, p = 0, \ \text{on } \partial\mathfrak{D}_t$$

is used to build up the regularity. Roughly speaking, one derivative of v in the interior gives two derivatives of p, which gives a gain of one time derivative of v since $D_t v + \partial p = 0$.

This gain of regularity in v also gives that for p and the free surface since it is the level surface of p (This is the key idea for the free surface problem for fluids). However, for the problem we study here, one has

$$\mathrm{div}(\mathscr{T}\nabla p) = -\partial_i v^j \partial_j v^i - D_t \mathrm{div} v, \tag{0.9}$$

$$D_t \mathrm{div} v = \kappa^2 \left((\Delta \mathscr{T})^2 + \mathscr{T} \Delta^2 \mathscr{T} + 2\partial \mathscr{T} \cdot \Delta \partial \mathscr{T} \right) \tag{0.10}$$

$$- \kappa \left(\Delta v^k \partial_k \mathscr{T} + 2\partial v^k \cdot \partial \partial_k \mathscr{T} \right), \tag{0.11}$$

$$p = 0, \text{ on } \partial \mathfrak{D}_t, \tag{0.12}$$

The above Dirichlet problem is used to build up the boundary regularity (The most challenging part for fluids free boundary problems is that the boundary regularity enters in the higher order energy estimates)

Equation for $\mathrm{div} v$:

$$D_t \mathrm{div} v - \kappa \mathscr{T} \Delta \mathrm{div} v = (\mathrm{div} v)^2 + 2\kappa \partial \mathscr{T} \cdot \partial \mathrm{div} v$$
$$- \kappa \left(\Delta v^k \partial_k \mathscr{T} + 2 \partial v^k \cdot \partial \partial_k \mathscr{T} \right)$$

$$\mathscr{T} = \mathscr{T}_b + \kappa \Delta^{-1} \mathrm{div} v,$$

where Δ^{-1} is for zero Dirichlet boundary condition.

Main Point: Coupling of Boundary Geometry and Interior Solutions

Two Geometric Quantities on the Free Surface

Orthogonal projection Π to the tangent space of the boundary: For a $(0,r)$ tensor $\alpha(t,x)$,

$$(\Pi\alpha)_{i_1\cdots i_r} = \Pi_{i_1}^{j_1}\cdots\Pi_{i_r}^{j_r}\alpha_{j_1\cdots j_r}, \quad \text{where} \quad \Pi_i^j = \delta_i^j - \mathcal{N}_i\mathcal{N}^j. \tag{0.13}$$

The tangential derivative of the boundary:

$$\bar{\partial}_i = \Pi_i^j\partial_j,$$

The second fundamental form of the boundary:

$$\theta_{ij} = \bar{\partial}_i\mathcal{N}_j.$$

A projection formula:

$$\Pi(\partial^2 q) = \theta \partial_{\mathcal{N}} q \quad \text{on} \quad \partial \mathfrak{D}_t,$$

if $q = const$ on $\partial \mathfrak{D}_t$, and its higher order version:

$$\Pi(\partial^r q) = \bar{\partial}^{r-2} \theta \partial_{\mathcal{N}} q + O(\partial^{r-1} q) + O(\bar{\partial}^{r-3} \theta) \text{ on } \partial \mathfrak{D}_t.$$

(This will be applied to $q = p$ and $q = D_t^r p$).

Injectivity radius

The injectivity radius of the normal exponential map of the boundary $\partial\mathfrak{D}_t$, $\imath_0$, is the largest number such that the map

$$\begin{aligned} \partial\mathfrak{D}_t \times (-\imath_0, \imath_0) \;\to\; \{x \in \mathbb{R}^n : \ \mathrm{dist}(x, \partial\mathfrak{D}_t) < \imath_0\} : \\ (\bar{x}, \imath) \;\mapsto\; x = \bar{x} + \imath \mathscr{N}(\bar{x}) \end{aligned}$$

is an injection.

Higher order energy functional

1. Construct a higher order energy functional from which the Sobolev norms of $H^s(\mathfrak{D}_t)$ ($\mathbb{N} \ni s \leq n+2$) of solutions will be derived.
2. Derive the bounds for some geometric quantities of the free surface such as the bound for the second fundamental form and lower bound for the injectivity radius of the normal exponential map. Those two parts are bonded together.

Construction of the higher order energy functional

We set $\kappa = 1$ from now on for the simplicity of the presentation. The energy functionals of each order are then defined by

$$E_0(t) = \int_{\mathfrak{D}_t} \mathscr{T}^{-1} |v|^2 dx, \tag{0.14a}$$

$$E_r(t) = \int_{\mathfrak{D}_t} \mathscr{T}^{-1} \delta^{mn} Q(\partial^r v_m, \partial^r v_n) dx \tag{0.14b}$$

$$+ \int_{\mathfrak{D}_t} |\partial^{r-1} \mathrm{curl}\, v|^2 dx + \int_{\mathfrak{D}_t} |\partial D_t^{r-1} \mathrm{div}\, v|^2 dx$$

$$+ \int_{\partial \mathfrak{D}_t} Q(\partial^r p, \partial^r p)(-\partial_{\mathscr{N}} p)^{-1} dS, \quad r \geq 1, \tag{0.14c}$$

where $D_t = \partial_t + v^k \partial_k$. The higher order energy functional is defined by

$$E(t) = \sum_{r=0}^{n+2} E_r(t)$$

.

Note that

$$\int_{\mathfrak{D}_t} |\partial^r v|^2 dx$$
$$\leq C(K) \int_{\mathfrak{D}_t} \delta^{mn} Q(\partial^r v_m, \partial^r v_n) dx$$
$$+ \int_{\mathfrak{D}_t} |\partial^{r-1} \mathrm{curl} v|^2 dx + \int_{\mathfrak{D}_t} |\partial^{r-1} \mathrm{div} v|^2 dx,$$

where

$$K = \max_{\partial \mathfrak{D}_t} (|\theta| + \iota^{-1}).$$

However, it does not work to control the time evolution of $\int_{\mathfrak{D}_t} |\partial^{r-1} \mathrm{div} v|^2 dx$. Instead, we control $\int_{\mathfrak{D}_t} |\partial D_t^{r-1} \mathrm{div} v|^2 dx$. After this, we can control $\int_{\mathfrak{D}_t} |\partial^{r-1} \mathrm{div} v|^2 dx$.

Quadratic form Q

Q is a positive definite quadratic form, such that for $(0, r)$ tensors α and β

1) $Q(\alpha, \beta) = < \Pi\alpha, \Pi\beta >$, is the inner product of the tangential components when it is restricted on the boundary,

2) in the interior $Q(\alpha, \alpha)$ increases to the norm $|\alpha|^2$ in the interior.

Here Π is the orthogonal projection to the tangent space of the boundary.

The construction of the above higher order energy functional is motivated by Chistodoulou and Lindblad's work on the problem for incompressible Euler equations.

Main Result

Initial Data Set

$$\mathrm{Vol}\mathfrak{D}_0 = \int_{\mathfrak{D}_0} dx, \quad K_0 = \max_{x \in \partial\mathfrak{D}_0} \{|\theta(0,x)| + |\iota_0^{-1}(0,x)|\}, \tag{0.15}$$

$$\underline{\varepsilon}_0 = \min_{x \in \partial\mathfrak{D}_0} (-\partial_{\mathcal{N}} p)(0,x), \tag{0.16}$$

$$\underline{\mathscr{T}}_0 = \min_{x \in \mathfrak{D}_0} \mathscr{T}(0,x), \quad \overline{\mathscr{T}}_0 = \max_{x \in \mathfrak{D}_0} \mathscr{T}(0,x), \tag{0.17}$$

$$M_0 = \max_{x \in \mathfrak{D}_0} \{|\partial p(0,x)| + |\partial v(0,x)| + |\partial \mathscr{T}(0,x)|\}. \tag{0.18}$$

The initial pressure $p_0(x) = p(0,x)$ is determined by solving a Dirichlet problem.

Main Theorem

Theorem

(L-H. H. Zeng) (ARMA 2021) Let $n = 2, 3$. Suppose that

$$0 < \mathrm{Vol}\mathfrak{D}_0,\ \underline{\varepsilon}_0,\ \underline{\mathscr{T}}_0,\ \overline{\mathscr{T}}_0 < \infty,\ K_0,\ M_0 < \infty.$$

Then there are continuous functions $\mathfrak{T}$ such that if

$$T \leq \mathfrak{T}\left(\mathrm{Vol}\mathfrak{D}_0, K_0, \underline{\varepsilon}_0^{-1}, \underline{\mathscr{T}}_0^{-1}, \overline{\mathscr{T}}_0, M_0, E_0(0), \cdots, E_{n+2}(0)\right),$$

then any smooth solution for $0 \leq t \leq T$ satisfies

$$\sum_{s=0}^{n+2} E_s(t) \le 2\sum_{s=0}^{n+2} E_s(0), \quad 0 \le t \le T, \tag{0.19a}$$

$$|\theta| + |\iota_0^{-1}| \le CK_0 \ \text{on} \ \partial\mathfrak{D}_t, \quad 0 \le t \le T, \tag{0.19b}$$

$$-\partial_{\mathcal{N}} p \ge 2^{-1}\underline{\varepsilon}_0 \ \text{on} \ \partial\mathfrak{D}_t, \quad 0 \le t \le T, \tag{0.19c}$$

Some Remarks

The bound for second fundamental form θ for free surfaces.

The projection formula,

$$\Pi(\partial^2 p) = \theta \partial_{\mathcal{N}} p \quad \text{on} \quad \partial \mathfrak{D}_t, \tag{0.20}$$

was used to estimate the L^∞-bound for θ in C-L (Christodoulou-Lindblad 2000). The reason that this can work in C-L is because one may obtain the L^∞-bound for $\partial^2 p$ on $\partial \mathfrak{D}_t$ independent of the L^∞-bound for θ, which, together with the lower bound for $-\partial_{\mathcal{N}} p$ due to the Taylor sign condition, gives the L^∞-bound for θ.

Indeed, it was proved in C-L that, for $n=2,3$,

$$\|\partial^2 p\|_{L^\infty(\partial\mathfrak{D}_t)} \le C(K_1)\sum_{r=2}^{n+1}\|\partial^r p\|_{L^2(\partial\mathfrak{D}_t)} \le C(K_1,\mathscr{E}_0,\cdots,\mathscr{E}_{n+1},\mathrm{Vol}\mathfrak{D}_t),$$

where

$$\mathscr{E}_0 = \int_{\mathfrak{D}_t}|v|^2 dx,$$

$$\mathscr{E}_r(t) = \int_{\mathfrak{D}_t}\Big(\delta^{mn}Q(\partial^r v_m,\partial^r v_n)+|\partial^{r-1}\mathrm{curl} v|^2\Big)dx$$

$$+\int_{\partial\mathfrak{D}_t}Q(\partial^r p,\partial^r p)(-\partial_{\mathscr{N}} p)^{-1}dS,\ r\ge 1,$$

K_1 is the upper bound for $1/\iota_1$ on $\partial\mathfrak{D}_t$ with ι_1 being another radius to measure the curvature of the boundary defined as: Let $0 < \varepsilon_1 \leq 1/2$ be a fixed number, and let $\iota_1 = \iota_1(\varepsilon_1)$ be the largest number such that

$$|\mathscr{N}(\bar{x}_1) - \mathscr{N}(\bar{x}_2)| \leq \varepsilon_1 \quad \text{whenever} \quad |\bar{x}_1 - \bar{x}_2| \leq \iota_1, \quad \bar{x}_1, \bar{x}_2 \in \partial\mathfrak{D}_t.$$

However, we can only obtain, for problem (0.4)-(0.6) that

$$
\begin{aligned}
&\|\partial^2 p\|_{L^\infty(\partial\mathfrak{D}_t)} \\
&\le C(K_1, \mathrm{Vol}\mathfrak{D}_t)\|\mathscr{T}^{-1}\|_{L^\infty(\mathfrak{D}_t)}\|\partial p\|_{L^\infty(\mathfrak{D}_t)}\|\theta\|_{L^\infty(\partial\mathfrak{D}_t)}\|\partial^n\mathscr{T}\|_{L^2(\partial\mathfrak{D}_t)} \\
&+ \text{other terms}, \quad n = 2, 3,
\end{aligned}
$$

from which it is clear that the projection formula used in C-L to give the L^∞-bound for θ cannot work directly for our problem.

Instead of using the projection formula, we use the evolution equations for θ. By doing so, we are led to the following estimate:

$$|D_t\theta| \leq |\partial^2 v| + C|\theta||\partial v|,$$

from which it is clear that we need to get the L^∞-bounds for both ∂v and $\partial^2 v$ on $\partial\mathfrak{D}_t$, while only the L^∞-bound for ∂v was sufficient in C-L. Thus, the L^∞-bound for one more derivative of the velocity field than that in C-L is needed. This causes the loss of one more derivative than that in C-L.

It should be noted that only ∂v enters equations (0.4), but not $\partial^2 v$, and thus one may think that the estimate of ∂v may be sufficient to close the argument as done in C-L. But the above argument suggests that this is not the case for the problem (0.4)-(0.6) which reflects the subtlety of this problem .
It is quite involved to bound $\partial^2 v$ before one obtains the L^∞-bound for θ in our case.

Thank You