Sydney University Mathematical Society

Problems Competition 1999

This competition is open to all undergraduates at any Australian university or tertiary institution. Prizes ($40 book vouchers from the Co-op Bookshop) will be awarded for the best correct solution to each of the 10 problems. Entrants from the University of Sydney will also be eligible for the Norbert Quirk Prizes (one for each of 1st, 2nd and 3rd years).

Contestants may use any source of information except other people. Solutions are to be received by 4.00 pm on Friday, September 10, 1999. They may be given to Dr. Donald Cartwright, Room 620, Carslaw Building, or posted to him at the School of Mathematics and Statistics, The University of Sydney, N.S.W. 2006. Entries must state name, university, student number, course and year, term address and telephone number, and be marked **1999 SUMS Competition**. The prizes will be awarded towards the end of the academic year.

The SUMS committee is grateful to all those who have provided problems. We are always keen to get more. Send any, with solutions, to Dr. Cartwright, at the above address.

Problems

(Extensions and generalizations of any problem are invited and are taken into account when assessing solutions.)

1. Suppose that you have 10 coins, and that you put them into piles, lined up so that the biggest pile is on the left, then the next biggest, and so on. Pick up all the coins from the biggest pile, and starting placing one of them on each of the remaining piles, starting from the left. Continue until the coins of the pile have been exhausted or, if you run out of piles before running out of coins, make as many new piles of size 1 as you need to use up the coins. Notice that if you start with piles of size 4,3,2 and 1, then this operation results in the same configuration of piles. Show that with any starting configuration, you will eventually reach the 4,3,2,1 configuration by repeating the above operation often enough.

2. An astronomer added up all the distances between 50 stars that he observed with a telescope, and got the answer S. Suddenly, a cloud obscured 25 of the stars. Show that the sum of the distances between the remaining 25 stars is less than $S/2$.

3. Let S be the set of complex numbers of the form $z_1 + z_2 + z_3$, where $|z_1| = |z_2| = |z_3| = 1$ and $z_1 z_2 z_3 = 1$. Describe S.

4. For any integer $n \geq 1$, $n(1+n)$ is divisible by $(0+1)(1+1) = 2$. In particular, this holds if n is a prime number. Find all pairs (ℓ, m) of non-negative integers with the property that $(\ell + p)(m + p)$ is divisible by $(\ell + 1)(m + 1)$ for all but a finite number of prime numbers p. Can you solve the analogous problem with triples (ℓ, m, n) of numbers?

5. Let E be a bounded subset of the real line which is a disjoint union of intervals. Let $|E|$ denote the total length of these intervals. Show that for any integer $n \geq 2$, it is possible to pick $n+1$ points $a_0, \ldots, a_n$ of E such that

$$\prod_{\substack{j=0 \\ j \neq \ell}}^{n} |a_j - a_\ell| \geq \left(\frac{|E|}{2e} \right)^n$$

for $\ell = 0, \ldots, n$.

6. Consider two triangles in the plane, with vertices A_1, A_2, A_3, and B_1, B_2, B_3, respectively. Show that the product of their areas is

$$-\frac{1}{16} \det \begin{pmatrix} d_{1,1}^2 & d_{1,2}^2 & d_{1,3}^2 & 1 \\ d_{2,1}^2 & d_{2,2}^2 & d_{2,3}^2 & 1 \\ d_{3,1}^2 & d_{3,2}^2 & d_{3,3}^2 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

where $d_{i,j}$ denotes the distance from A_i to B_j.

7. Suppose that $A = (a_{i,j})$ is an $n \times n$ matrix of 0's and 1's such that there is no 2×2 submatrix

$$\begin{pmatrix} a_{i,r} & a_{i,s} \\ a_{j,r} & a_{j,s} \end{pmatrix}$$

(where $1 \leq i < j \leq n$ and $1 \leq r < s \leq n$) all of whose entries are 1. Show that the number of 1's in A is at most $n\sqrt{2n-1}$.

8. Let $n \geq 1$ be an integer. Consider the polynomials of the form

$$(1 - t^{n_1})(1 - t^{n_2}) \cdots (1 - t^{n_r}),$$

where $n_1, \ldots, n_r$ are positive integers adding to n. Find the least common multiple of these polynomials. For example, if $n = 4$, then the polynomials are

$$(1-t)^4,\ (1-t)^2(1-t^2),\ (1-t)(1-t^3),\ (1-t^2)^2,\ (1-t^4).$$

9. Suppose that a function $f(x, y)$ has second derivatives f_{xx} and f_{yy} at each point of a disc about a point (x_0, y_0), and suppose that f_{xx} and f_{yy} are continuous at the point (x_0, y_0). Suppose also that the two mixed derivatives f_{xy} and f_{yx} exist at (x_0, y_0) but are not necessarily continuous there. Prove that $f_{yx} = f_{xy}$ at the point (x_0, y_0).

10. Find a formula for the number of permutations of $1, \ldots, n$ such that 1 comes before 2, 2 after 3, 3 before 4, etc.