Sydney University Mathematical Society Problem Competition 2012

This competition is open to undergraduates (including Honours students) at any Australian university or tertiary institution. Those who enter do so as individuals, and must **not** receive help with the problems, e.g. from fellow students, lecturers or online groups.

Entrants may submit solutions to as many problems as they wish. Prizes ($60 book vouchers from the Co-op Bookshop) will be awarded for the best solution to each of the 10 problems. Students from the University of Sydney are also eligible for the Norbert Quirk Prizes, based on the overall quality of their entry (one for each of 1st, 2nd and 3rd years).

Extensions and generalizations of any problem are invited and are taken into account when assessing solutions. If two or more solutions to a problem are essentially equal, the prize will be given to the student(s) in the earlier year of university. If a problem receives no complete solutions, partial solutions may suffice.

Entries must be received by **Wednesday, August 8, 2012**. They may be posted to Associate Professor Anthony Henderson, School of Mathematics and Statistics, The University of Sydney, NSW 2006, or handed in to Carslaw room 520. Please mark your entry SUMS Problem Competition 2012, and include your name, university, student number, year of study, and postal address (or email address for University of Sydney students) for the awarding of prizes. Please note that entries will not be returned.

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1. Alice and Bess are playing a game where Alice thinks of a number in the set $A = \{1, 2, 3, 4, 5, 6\}$ and Bess has to guess what it is. If she guesses correctly, she wins; if she guesses incorrectly, Alice increases or decreases her number by 1 (keeping it in the set A) before Bess' next guess. What is the smallest number k such that Bess can guarantee to win within k guesses?

2. Show that there exists an infinite set X of points in the plane such that no three points in X lie on a line, and the distance between any two points in X is a rational number.

3. Find the largest positive real number α for which the sequence $\left(1 + \frac{\alpha}{n}\right)^{n+1}$ (for $n = 1, 2, 3, \cdots$) is monotonically decreasing.

4. In this problem, a *word* means a string of letters drawn from the three-letter alphabet A, B, C. Say that a word is *decent* if it does not contain two consecutive identical letters, and also does not contain AB as a consecutive substring. Find the number of decent words of length n.

5. In this problem, S denotes a subset of the set of real numbers.

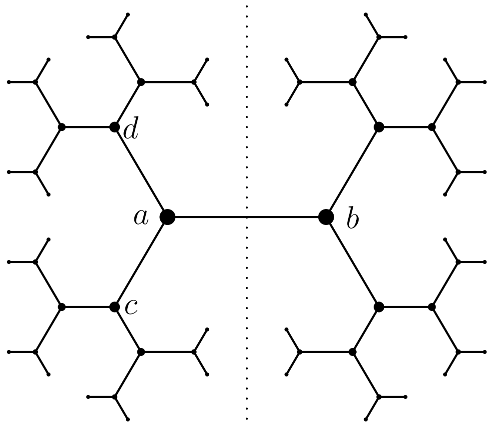
a) Suppose that $1 \in S$, S is closed under subtraction in the sense that $a, b \in S \Rightarrow a - b \in S$, and S is closed under taking inverses in the sense that $0 \neq a \in S \Rightarrow a^{-1} \in S$. Prove that S is closed under multiplication in the sense that $a, b \in S \Rightarrow ab \in S$.

b) Show that the assumption $1 \in S$ in a) is necessary: that is, give an example of an S that is closed under subtraction and taking inverses, but not closed under multiplication.

6. Let x be a positive real number. Define a sequence $(a_0(x), a_1(x), a_2(x), \cdots)$ by the initial condition $a_0(x) = x$ and the recursion $a_n(x) = \dfrac{a_{n-1}(x)^2}{n}$ for all $n \geq 1$. For which x does this sequence converge?

7. A *tree* is a connected simple graph with no cycles. For a tree T, let $s(T)$ denote the number of nonempty subsets X of the set of vertices of T such that for any two vertices in X, there is a path in T joining them that only passes through vertices in X. For a positive integer n, find the minimum and maximum values of $s(T)$ as T ranges over all trees with n vertices.

8. In this problem, let T denote a 3-regular tree ("3-regular" means that every vertex is adjacent to 3 others). The vertex set of T is infinite, but this picture gives an indication of part of it:



As shown here, T can be embedded in the plane so that the edges at each vertex are at angles of $120°$, and the whole tree is symmetric under reflection in the dotted line. That reflection σ is one example of an *automorphism* of T (a permutation of the vertices under which adjacent vertices map to adjacent vertices). Another is the 'rotation' ρ, which fixes a, sends b to c, c to d, and d to b, and rotates the direction of each edge by $120°$ clockwise, though it does not preserve the lengths of edges. An *allowable* automorphism of T is one that may be obtained by repeatedly performing ρ and σ in some order. Show that for any two vertices v and w of T, there are exactly three allowable automorphisms that send v to w.

9. For a permutation σ of $\{1, 2, 3, \cdots, n\}$, a *break* of σ is an element k of $\{1, 2, \cdots, n-1\}$ such that $\sigma(\{1, \cdots, k\}) = \{1, \cdots, k\}$. The *score* of σ is the square of the number of breaks. Show that the average score of all permutations of $\{1, 2, 3, \cdots, n\}$ tends to 0 as n tends to infinity.

10. Let S denote the polynomial ring $\mathbb{C}[x_1, x_2, x_3, \cdots]$. Define a linear operator Δ on S by

$$\Delta(p) = \sum_{r \geq 0} \left(\sum_{m_1 + 2m_2 + \cdots + rm_r = r} \frac{x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r}}{1^{m_1} m_1!\, 2^{m_2} m_2! \, \cdots \, r^{m_r} m_r!} \right) \frac{\partial p}{\partial x_{r+1}}.$$

Here the outer sum, over nonnegative integers r, makes sense because each $p \in S$ involves only finitely many of the variables, so $\frac{\partial p}{\partial x_{r+1}} = 0$ for sufficiently large r. The inner sum is over all r-tuples $(m_1, m_2, \cdots, m_r)$ of nonnegative integers satisfying the stated condition $m_1 + 2m_2 + \cdots + rm_r = r$. (There is an empty 0-tuple, so the $r = 0$ term is $\frac{\partial p}{\partial x_1}$.)

a) For each integer $k \geq 2$, let $p_k = 2(k-1)x_k - \displaystyle\sum_{i=1}^{k-1} x_i x_{k-i}$. Show that $\Delta(p_k) = 0$.

b) Show that the kernel of Δ consists exactly of the polynomials in $p_2, p_3, p_4, \cdots$.