Sydney University Mathematical Society Problem Competition 2015

This competition is open to undergraduates (including Honours students) at any Australian university or tertiary institution. Those who enter do so as individuals, and must **not** receive help with the problems, e.g. from fellow students, lecturers or online groups.

For each of the eight problems, a $75 voucher from the Co-op will be awarded for the best solution. Entrants may submit solutions to whichever problems they wish. Students at the University of Sydney are also eligible for the Norbert Quirk Prizes, based on the overall quality of their entry; there is one Norbert Quirk Prize for each of 1st, 2nd, 3rd and Honours years.

The problems are listed in roughly increasing order of difficulty. For the easier problems, solutions which include an interesting extension or generalization will be rated more highly. For the harder problems, partial solutions may be accepted. If two or more solutions to a problem are essentially equal, the prize will be given to the student(s) in the earlier year of university.

Entries must be received on or before **Friday, August 7, 2015**. This year, for the first time, all entries must be sent by email to Associate Professor Anthony Henderson at:

`anthony.henderson@sydney.edu.au`

Entries may either be an electronic file (pdf preferred) or a legible scan of handwriting. Please mark your entry SUMS Problem Competition 2015, and include your name, university, student number, year of study, and postal address (or email address for University of Sydney students) for the awarding of prizes.

---

1. Let $\mathbb{Z}^+$ denote the set of positive integers. If $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ is a function and $m \in \mathbb{Z}^+$, let $f^{(m)}$ denote the composite function $f \circ f \circ \cdots \circ f$ (with m copies of f). Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ with the property that $f^{(m)}(n) = f(mn)$ for all $m, n \in \mathbb{Z}^+$.

2. Let n be a positive integer. Prove the inequality

$$\sum_{k=1}^{n} \sqrt{n^2 - k^2}\sqrt{n^2 - (k-1)^2} \; < \; \frac{2n^3 + n}{3}.$$

3. Let n be a positive integer. A *composition* of n is an ordered k-tuple $(n_1, n_2, \cdots, n_k)$ of positive integers satisfying $n_1 + n_2 + \cdots + n_k = n$. Let $\mathcal{C}(n)$ be the set of all compositions of n, where the length k of the tuple is allowed to vary (it can be anything from 1 to n). Prove that

$$\sum_{(n_1, n_2, \cdots, n_k) \in \mathcal{C}(n)} (-1)^{n-k}\, 1^{n_1} 2^{n_2} \cdots k^{n_k} \; = \; 1.$$

4. If P is a convex polygon in the plane, let $M(P)$ be the convex polygon whose vertices are the midpoints of the edges of P. Say that P is *periodic* if $M^k(P)$ is similar to P for some positive integer k, where M^k denotes k applications of the operation M. For example, every triangle T is periodic, because $M(T)$ is similar to T; every parallelogram Q is periodic, because $M^2(Q)$ is similar to Q. Show that there is a periodic pentagon in which no two edges have the same length.

5. Let F be the field of integers modulo p, where p is a prime number. Define a finite set

$$X = \{(x, y, z) \in F^3 \mid x^6 + y^3 + z^2 = 0\}.$$

Show that $|X| = p^2$ if and only if $p \not\equiv 1 \pmod 6$.

6. Define a function $f : (-\infty, 1) \to \mathbb{R}$ by

$$f(x) = \int_0^1 \frac{\sqrt{2-x}}{\sqrt{1-s^2}\sqrt{1-xs^2}}\, ds.$$

Show that $f(x)$ has a global minimum at $x = 0$.

7. Let $\zeta = e^{\pi i/6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$, and let $\mathbf{Z}[\zeta]$ denote the set of integer linear combinations of the powers of ζ. Suppose that $u, v \in \mathbf{Z}[\zeta]$ satisfy $|u|^2 = \sqrt{3}\,|v|^2 + 1$ and $v \neq 0$. Show that $|v|^2 \geq 2 + \sqrt{3}$, and find when equality occurs.

8. Let d be a fixed integer, at least 2. If $P(x)$ is a polynomial in x, let $\lceil P(x) \rceil$ be the polynomial obtained by rounding up each exponent of x to the nearest multiple of d, so that $\lceil P(x) \rceil$ is a polynomial in x^d. For example, if $d = 3$ then

$$\lceil 2 + 5x^2 + 4x^3 + x^4 \rceil = 2 + 5x^3 + 4x^3 + x^6 = 2 + 9x^3 + x^6.$$

Suppose that all we know about $P(x)$ is that it has nonnegative real coefficients. Show that if we are given all of the polynomials $\lceil P(x) \rceil$, $\lceil P(x)^2 \rceil$, $\lceil P(x)^3 \rceil$, $\ldots$, we can determine $P(x)$.