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COVOLUMES OF UNIFORM LATTICES ACTING ON POLYHEDRAL COMPLEXES

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Abstract

Let X be a polyhedral complex with finitely many isometry classes of links. We establish a restriction on the covolumes of uniform lattices acting on X. When X is two-dimensional and has all links isometric to either a complete bipartite graph or the building for a Chevalley group of rank 2 over a field of prime order, we obtain further restrictions on covolumes.

1. *Introduction*

Let G be a locally compact topological group, with suitably normalised Haar measure μ. Let $\Gamma \leq G$ be a *uniform lattice*, that is, a cocompact discrete subgroup. A basic question is: which covolumes $\mu(\Gamma\backslash G)$ can occur?

When G is a Lie group, either real or p-adic, this is a well-studied classical problem. See, for example, [**23**], [**21**] and [**19**]. A non-classical case is that of the automorphism group of a locally finite tree, which is naturally a locally compact topological group; see [**2**]. Levich and Rosenberg ([**22**], Chapter 9) have completely classified the covolumes of uniform lattices acting on regular and biregular trees.

In this note, we study the higher-dimensional case of covolumes of uniform lattices acting on locally finite n-dimensional polyhedral complexes, for $n \geq 2$. Some of these complexes are generalisations of buildings. Their automorphism groups are locally compact when endowed with a natural topology, and the corresponding Haar measure may be suitably normalised. See Section 2 for precise definitions and this normalisation.

The main result obtained is Theorem 2 below. This theorem applies to a large class of polyhedral complexes, those where the quotient by the full automorphism group is finite, so that there are finitely many isometry classes of links (the link of a vertex of a polyhedral complex is defined in Section 2.1). If X is such a complex, and $\Gamma \leq G = \mathrm{Aut}(X)$ is a uniform lattice, Theorem 2 may be summarised as:

Summary of Theorem 2. *There is a restriction on $\mu(\Gamma\backslash G)$ based only on the prime divisors of the orders of the automorphism groups of the finite simplicial complexes which are the links of X.*

The statement and proof of Theorem 2 are found in Section 3.1, and in Section 3.2 we apply Theorem 2 to many examples, including some in dimensions greater than 2. In Section 4, we use an analogue of Theorem 2 to establish even stronger restrictions on covolumes of uniform lattices which act on certain 2-dimensional complexes

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with only *one* isometry class of link. For example, we apply our general result to "Bourdon buildings" (see Bourdon [**4**] and Bourdon–Pajot [**6**], [**7**]) to obtain:

COROLLARY. *Let $L = K_{m,n}$ be a complete bipartite graph, with $m \geq n \geq 2$, and let $r \geq 5$, with r even if $m \neq n$. Let P be a regular right-angled hyperbolic r-gon. Suppose X is the (unique) polygonal complex with all links isometric to L and all 2-cells isometric to P. Then if Γ is a uniform lattice in $G = \mathrm{Aut}(X)$,*

$$\mu(\Gamma\backslash G) = \frac{r}{mn}\left(\frac{a}{b}\right)$$

where a/b is rational (in lowest terms), and the prime divisors of b are strictly less than m.

In fact, this result is sharp, in the sense that, as we show in [**26**], every possible covolume satisfying the restrictions of this corollary actually occurs for some uniform lattice $\Gamma \leq G$.

2. *Background*

In this section we give the basic definitions of polyhedral complexes, their automorphism groups and lattices, and describe several methods for constructing these complexes and lattices.

2.1. *Polyhedral complexes*

Let $\mathbb{X}^n$ be S^n, $\mathbb{R}^n$ or $\mathbb{H}^n$, endowed with metrics of constant curvature 1, 0 and -1 respectively. A *polyhedral complex* X is a CW-complex such that:

(1) each open cell of dimension n is isometric to the interior of a compact convex polyhedron in $\mathbb{X}^n$; and
(2) for each cell σ of X, the restriction of the attaching map to each open codimension 1 face of σ is an isometry onto an open cell of X.

A polyhedral complex is said to be (piecewise) *spherical*, *Euclidean* or *hyperbolic* if $\mathbb{X}^n$ is S^n, $\mathbb{R}^n$ or $\mathbb{H}^n$ respectively. A 2-dimensional polyhedral complex is called a polygonal complex.

Let x be a vertex of a d-dimensional polyhedral complex X. The *link* of x, written $\mathrm{Lk}(x, X)$, is the spherical simplicial complex obtained by intersecting X with a d-sphere of sufficiently small radius centred at the vertex x. For example, if X has dimension 2, then $\mathrm{Lk}(x, X)$ may be identified with the graph having vertices the 1-cells of X containing x and edges the 2-cells of X containing x; an edge joins two vertices in the link if the corresponding 2-cell contains those two 1-cells. By rescaling so that for each x the d-sphere around x has radius say 1, we induce a metric on each link, and we may then speak of isometry classes of links of X.

There are several constructions of polyhedral complexes with specified links.

(1) Buildings of dimension n have cells which are Euclidean simplexes, and the links of all vertices isometric to a spherical building of dimension $(n-1)$.
(2) The Davis–Moussong complex [**11**] is constructed from a metric flag complex L. There exists a CAT(0) polyhedral complex X_L such that the link of every vertex in X_L is isometric to L.
(3) Ballmann–Brin [**1**], by attaching edges and polygons step-by-step, have constructed polygonal complexes with all cells regular Euclidean k-gons, $k \geq 6$,

and all links the 1-skeleton of either the n-simplex, the n-cube or a Platonic solid.

(4) Polyhedral complexes may be realised as universal covers of complexes of groups (see [**24**], [**10**], [**13**] and [**8**] for the theory of complexes of groups). In this way, Haglund [**14**] has obtained similar results to [**1**], while Benakli has constructed polygonal complexes so that every link is isometric to one of finitely many given graphs [**3**]. Bourdon has constructed so-called *Fuchsian buildings*, which are hyperbolic polygonal complexes, using complexes of groups [**5**].

(5) Haglund–Paulin [**16**] have constructed 3-dimensional polyhedral complexes with each 3-cell a hyperbolic polytope and the link of each vertex the flag complex of projective 3-space over a finite field. They used decompositions of buildings, which reflect decompositions of their Coxeter systems.

2.2. *Lattices and covolumes*

Let G be a locally compact topological group with left-invariant Haar measure μ. A discrete subgroup $\Gamma \leq G$ is called a *lattice* if the covolume $\mu(\Gamma\backslash G)$ is finite. A lattice Γ is *uniform* if $\Gamma\backslash G$ is compact. Let S be a left G-set such that for every $s \in S$, the stabiliser G_s is compact and open. Then if $\Gamma \leq G$ is discrete, the stabilisers Γ_s are finite. We then define the S-*covolume* of Γ by

$$\mathrm{Vol}(\Gamma\backslash\backslash S) = \sum_{s\in\Gamma\backslash S} \frac{1}{|\Gamma_s|} \leq \infty$$

The following theorem shows that Haar measure may be normalised so that $\mu(\Gamma\backslash G)$ equals the S-covolume.

THEOREM 1 ([**2**], Chapter 1). *Let G be a locally compact topological group acting on a set S with compact open stabilisers and a finite quotient $G\backslash S$. Suppose further that G admits at least one lattice. Then there is a normalisation of the Haar measure μ, depending only on the choice of G-set S, such that for each discrete subgroup Γ of G we have $\mu(\Gamma\backslash G) = \mathrm{Vol}(\Gamma\backslash\backslash S)$.*

Let X be a connected, locally finite, d-dimensional polyhedral complex, with vertex set $V(X)$. We write $\mathrm{Aut}(X)$ for the group of polyhedral isometries of X. A subgroup of $\mathrm{Aut}(X)$ is said to act *without inversions* if its elements fix pointwise each cell that they preserve. Let σ be a cell of X. For $n \geq 0$, we define *combinatorial balls* $B(\sigma, n)$ centred at σ by induction. The ball $B(\sigma, 0)$ is just the cell σ, while for $n \geq 1$, $B(\sigma, n)$ is the union of the d-cells of X which meet $B(\sigma, n-1)$.

The group $G = \mathrm{Aut}(X)$ naturally has the structure of a locally compact topological group, with a neighbourhood basis of the identity consisting of automorphisms fixing larger and larger combinatorial balls. A subgroup $\Gamma \leq G$ is discrete in this topology if and only if the stabiliser Γ_x is finite for each $x \in V(X)$. By the same arguments as for tree lattices ([**2**], Chapter 1), it can be shown that if $G\backslash X$ is finite, then a discrete subgroup $\Gamma \leq G$ is a uniform lattice if and only if its $V(X)$-covolume is a sum with finitely many terms. Finally, using Theorem 1, we now normalise the Haar measure μ on $G = \mathrm{Aut}(X)$ so that for all uniform lattices $\Gamma \leq G$, the covolume of Γ is

$$\mu(\Gamma\backslash G) = \mathrm{Vol}(\Gamma\backslash\backslash V(X))$$

There are several constructions of uniform lattices acting on polyhedral complexes. For buildings, there are arithmetic lattices, while for the Davis–Moussong complex, the Coxeter group W_L associated to L is a uniform lattice in $\mathrm{Aut}(X_L)$. If a polyhedral complex X is constructed as the universal cover of a (faithful) finite complex of finite groups, then the fundamental group of this complex of groups is a uniform lattice acting without inversions on X.

3. *Covolumes for finitely many link types*

In this section we state and prove Theorem 2, a very general restriction on covolumes of uniform lattices acting on complexes with finite fundamental domains, and thus finitely many isometry classes of links. Then in Section 3.2 we give some applications of this theorem.

3.1. *Covolume restrictions*

Theorem 2 below can be seen as a generalisation of the following result on tree lattices, due to Levich (in [**22**], Lemma 9.1.1). Let T_m be the m-regular tree, and suppose a/b, a rational in lowest terms, is the covolume of a uniform lattice $\Gamma \leq \mathrm{Aut}(T_m)$. Then b is not divisible by any primes greater than m, and if m is prime then b is not divisible by m^2 (such a b is called an m-number). The key step in the proof is showing that for all vertices v of the tree, the order of the stabiliser Γ_v is an m-number. For this, Γ_v is injected into the automorphism group of a finite rooted tree, and this rooted automorphism group is computed. In the proof of Theorem 2 we also inject the vertex stabiliser into a rooted automorphism group, but derive information about the order of the rooted group without actually computing the group itself.

Theorem 2. *Let X be a d-dimensional polyhedral complex such that $G\backslash X$ is finite, where $G = \mathrm{Aut}(X)$. Let $L_1, L_2, \ldots, L_m$ be the finite simplicial complexes such that for each $x \in V(X)$, $\mathrm{Lk}(x,X)$ is isometric to some L_i. Let*

$$p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_M^{\alpha_M}$$

be the lowest common multiple of the orders of the groups $\mathrm{Aut}(L_i)$, $1 \leq i \leq m$, with each p_j prime, $p_1 < p_2 < \ldots < p_M$, and $\alpha_j > 0$ for $1 \leq j \leq M$. For $1 \leq i \leq m$ and each $(d-1)$-cell σ of L_i, let $\mathrm{Fix}_i(\sigma)$ be the subgroup of $\mathrm{Aut}(L_i)$ which fixes that cell pointwise. Let

$$p_1^{\alpha'_1} p_2^{\alpha'_2} \ldots p_M^{\alpha'_M}$$

be the lowest common multiple of the orders of the groups $\mathrm{Fix}_i(\sigma)$, for $1 \leq i \leq m$, so that $0 \leq \alpha'_j \leq \alpha_j$ for $1 \leq j \leq M$. Then if Γ is a uniform lattice in G, its covolume $\mu(\Gamma\backslash G) = a/b$ is a rational in lowest terms, such that b is:
(a) not divisible by any primes other than $p_1, p_2, \ldots, p_M$; and
(b) if for some j we have $\alpha'_j = 0$, then not divisible by $p_j^{\alpha_j+1}$.

Proof. By definition the covolume of Γ is the finite sum

$$\mu(\Gamma\backslash G) = \sum_{x \in \Gamma\backslash V(X)} \frac{1}{|\Gamma_x|}$$

We will show that (a) and (b) hold for the order of each stabiliser Γ_x, and so complete the proof.

Since Γ_x is finite, for n sufficiently large Γ_x injects into the finite group $H_n := \mathrm{Aut}(B(x,n))$. We will prove by induction on $n \geq 1$ that (a) and (b) hold for the order of H_n. By Lagrange's Theorem, (a) and (b) then hold for the order of any subgroup of H_n.

To begin the induction, when $n = 1$, the ball $B(x,n)$ consists only of those d-cells corresponding to $(d-1)$-cells in $\mathrm{Lk}(x,X)$. Hence, we may identify H_1 with a subgroup of $\mathrm{Aut}(L_i)$ for some i. Thus (a) and (b) hold for the order of H_1 (irrespective of the values of the α'_j).

Assume (a) and (b) hold for $|H_n|$. An element of $H_{n+1} = \mathrm{Aut}(B(x,n+1))$ fixes x, and so restricts to an element of H_n. Let $\varphi : H_{n+1} \to H_n$ be this restriction homomorphism, with kernel K and image I, so that

$$|H_{n+1}| = |K||I|$$

By induction, since I is a subgroup of H_n, (a) and (b) hold for the order of I. We will show that (a) holds for $|K|$, and that if $\alpha'_j = 0$ then no power of p_j divides $|K|$.

Now, since $B(x,n)$ contains finitely many cells, finitely many vertices of X lie in the boundary of $B(x,n)$. Enumerate these boundary vertices as $x_1, x_2, \ldots, x_N$, and consider the restriction homomorphisms

$$K \to \mathrm{Aut}(B(x_k,1)), \text{ for } 1 \leq k \leq N$$

We may identify each $\mathrm{Aut}(B(x_k,1))$ with a subgroup of $\mathrm{Aut}_\ell(L_i)$, for some i. Since (a) holds for the order of a subgroup of any $\mathrm{Aut}_\ell(L_i)$, we have that (a) holds for the order of the image of this restriction map. Moreover, if K' is any subgroup of K, then (a) also holds for the order of the image of the restriction homomorphisms

$$K' \to \mathrm{Aut}(B(x_k,1)), \text{ for } 1 \leq k \leq N$$

For each k, at least one $(d-1)$-cell in the link of the vertex x_k corresponds to a d-cell of $B(x,n)$. So, as elements of K fix $B(x,n)$ pointwise, at least one $(d-1)$-cell in the link of each x_k is fixed pointwise by K. If for some j we have $\alpha'_j = 0$, then p_j does not divide the order of any of the subgroups $\mathrm{Fix}_\ell(e, L_i)$. So if $\alpha'_j = 0$, the order of the image of the restriction homomorphism $K \to \mathrm{Aut}(B(x_k,1))$ is not divisible by p_j. And if K' is any subgroup of K, then the order of the image of the restriction map $K' \to \mathrm{Aut}(B(x_k,1))$ will also not be divisible by p_j.

Put $K_0 = K$, and for $k = 1, 2, \ldots, N$ define K_k and I_k to be respectively the kernel and image of the restriction map

$$K_{k-1} \to \mathrm{Aut}(B(x_k,1))$$

Then

$$K_N \leq K_{N-1} \leq \ldots \leq K_2 \leq K_1 \leq K_0 = K$$

This implies that for all $1 \leq k \leq N$, (a) holds for the order of I_k, and if $\alpha'_j = 0$ then $|I_k|$ is not divisible by p_j. Since

$$|K| = |K_N||I_N||I_{N-1}| \ldots |I_2||I_1|$$

and K_N is trivial, we are done. □

3.2. *Examples*

We now apply Theorem 2 to some examples in dimensions 2, 3 and 4. The notation is as in the statement of the theorem, so that in each case $\mu(\Gamma\backslash G) = a/b$ is a rational in lowest terms.

(1) Let $L_i = K_{m_i,n_i}$ be a complete bipartite graph with $m_i \geq n_i \geq 2$, for $1 \leq i \leq m$. Let S_n be the symmetric group on n letters. Then if $m_i = n_i$,

$$\mathrm{Aut}(L_i) = (S_{m_i} \times S_{n_i}) \rtimes S_2$$

and if $m_i > n_i$,

$$\mathrm{Aut}(L_i) = S_{m_i} \times S_{n_i}$$

Theorem 2 implies that b is not divisible by any primes greater than $M = \max_i\{m_i\}$. The subgroup of $\mathrm{Aut}(L_i)$ which fixes an edge pointwise is $S_{m_i-1} \times S_{n_i-1}$. Hence, if M is prime and the graph $K_{M,M}$ is one of the links then b is not divisible by M^3, and if M is prime and $\max_i\{n_i\} < M$ then b is not divisible by M^2.

(2) Let $m = 1$ and $L_1 = L$ be the *Petersen graph* (see, for example, [**20**] pp. 240–241). The 10 vertices of this graph may be identified with the set of transpositions in S_5, and two vertices are joined by an edge if those transpositions are disjoint. Now

$$\mathrm{Aut}(L) = S_5$$

so b is not divisible by any primes greater than 5. The subgroup of $\mathrm{Aut}(L)$ which fixes an edge pointwise is $S_2 \times S_2$, so b is not divisible by 3^2 or 5^2.

(3) Let $m = 1$ and $L_1 = L$ be the flag complex of the projective plane over a finite field $\mathbb{F}_q$. The group $\mathrm{Aut}_0(L)$ of type-preserving automorphisms of L has index 2 in $\mathrm{Aut}(L)$, and is isomorphic to $P\Gamma L_3(\mathbb{F}_q)$, the group of incidence-preserving bijections of the projective plane over $\mathbb{F}_q$. If $q = p^n$, where p is prime, then the order of $P\Gamma L_3(\mathbb{F}_q)$ is given by (see, for example, [**17**] Theorem 2.8):

$$|P\Gamma L_3(\mathbb{F}_q)| = n|PGL_3(\mathbb{F}_q)| = nq^3(q^3-1)(q^2-1)$$

Hence, b is not divisible by any primes other than those dividing n, q, $(q^2 + q + 1)$ and $(q \pm 1)$. Since $\mathrm{Aut}_0(L)$ acts transitively on the $(q+1)(q^2+q+1)$ edges of L, the subgroup of $\mathrm{Aut}_0(L)$ which fixes an edge pointwise has order $nq^3(q-1)^2$. Depending on the value of q, this may tell us more about the prime divisors of b.

(4) More generally, let $\mathcal{G}$ be a finite, rank 2 Chevalley group over a finite field $\mathbb{F}_q$. Let $m = 1$ and $L_1 = L$ be the spherical building associated to the BN-pair of $\mathcal{G}$. The group $\mathrm{Aut}_0(L)$ is an extension of $\mathcal{G}$ by $\mathrm{Aut}(\mathbb{F}_q)$ ([**27**], Corollary 5.9). Table 1 of [**12**] gives the orders of the groups $\mathcal{G}$ and the number of edges of the corresponding graphs L. Since $\mathrm{Aut}_0(L)$ acts transitively on the set of edges of L, the order of a subgroup of $\mathrm{Aut}_0(L)$ which fixes an edge of L pointwise may thus be found. The previous example is the case where $\mathcal{G}$ is of type A_2.

(5) Similar arguments to Example (3) may be used for the 3-dimensional complexes constructed by Haglund–Paulin in [**16**] (see Example (5), Section 2.1), where the link is the flag complex of projective 3-space.

(6) Let $m = 1$ and $L_1 = L$ be the first barycentric subdivision of the 120-cell,

the regular polytope in $\mathbb{R}^4$ whose boundary consists of 120 dodecahedrons (see, for example, [**25**]). The Davis–Moussong complex associated to L is a 4-dimensional right-angled hyperbolic building ([**18**], proof of Theorem 2). The automorphism group of the 120-cell has order 120^2 [**9**], so b has no prime divisors other than 2, 3 and 5, and since the subgroup fixing a dodecahedron pointwise is trivial, b must be a factor of 120^2.

4. *Covolumes for polygonal complexes with one link type*

In the remainder of this note we consider 2-dimensional complexes. A polygonal complex is said to be an (r, L)*-complex* if all of its 2-cells are isometric to regular r-gons, and the links of all of its vertices are isometric to a graph L. In this section we establish restrictions stronger than those of Theorem 2 on covolumes of uniform lattices acting on certain (r, L)-complexes. First, in Sections 4.1 and 4.2 we recall results on the existence and uniqueness of these (r, L)-complexes, and constructions of uniform lattices acting on them. Section 4.3 then contains our restrictions on covolumes.

4.1. *Existence and uniqueness of* (r, L)*-complexes*

In general, there may be uncountably many pairwise non-isomorphic (r, L)-complexes (see [**1**] Theorem 1.6, [**14**], and [**12**] Theorem 3.6). In the following cases, however, local data does uniquely determine a polygonal complex.

(1) Let $L = K_{m,n}$ be a complete bipartite graph, for $m, n \geq 2$, and let $r \geq 5$, with r even if $m \neq n$. Let P be a regular right-angled hyperbolic r-gon. Then there exists a unique connected (r, L)-complex X such that all 2-cells of X are isometric to P. This is due to Bourdon ([**4**], Proposition 2.2.1) and Haglund ([**15**], Theorem 2).

(2) Let L be the spherical building associated to a finite Chevalley group $\mathcal{G}$ of rank 2 over a finite field $\mathbb{F}_q$. Then L is a generalised m-gon, for some $m \geq 3$. Let $r \geq 5$ and let P be a regular hyperbolic r-gon with all vertex angles $\frac{\pi}{m}$. A connected (r, L)-complex X with all 2-cells isometric to P is called an (r, L)*-building*. We say that X is *locally reflexive* if along each edge of X, the subcomplex consisting of the 2-cells meeting that edge possesses an automorphism of order 2 (see [**15**] for the exact definitions). Then if q is prime, and $r \geq 6$ is even, there exists a unique locally reflexive (r, L)-building X. This is due to Haglund ([**15**], Theorem 2).

4.2. *Uniform lattices acting on* (r, L)*-complexes*

For each of the examples in Section 4.1, where an (r, L)-complex is specified by local data, we describe a uniform lattice acting on that (r, L)-complex. The constructions (due to Bourdon [**5**] and Gaboriau–Paulin [**12**]) were originally complexes of groups. Here, we state the stabilisers of faces, edges and vertices in the quotient.

(1) Let X be as in (1) of Section 4.1. Then there is a uniform lattice $\Gamma \leq \mathrm{Aut}(X)$ so that the quotient $\Gamma\backslash X$ is the polygon P. The face stabiliser is trivial. If L is $K_{m,m}$ then the stabiliser of each edge is $\mathbb{Z}/m\mathbb{Z}$ and of each vertex is $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$. If L is $K_{m,n}$ with $m \neq n$, so that P has an even number of sides, then the edge stabilisers alternate between $\mathbb{Z}/m\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$, and

the vertex stabilisers are $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. This is a rephrasing of [**5**], Example 1.5(a).

(2) Let X be as in (2) of Section 4.1. Then there is a uniform lattice $\Gamma \leq \mathrm{Aut}(X)$ so that the quotient $\Gamma\backslash X$ is the polygon P. The face stabiliser is the group B of the BN-pair of the Chevalley group $\mathcal{G}$. The edge stabilisers alternate between P_1 and P_2, where P_1 and P_2 are parabolic subgroups of $\mathcal{G}$. The vertex stabilisers are the group $\mathcal{G}$. This is a rephrasing of [**12**], Section 3.1.4, Example (A).

4.3. *Covolume restrictions*

We conclude by establishing restrictions on covolumes of uniform lattices acting on the complexes described in Section 4.1. If X is as in (1) or (2) of Section 4.1, then $\mathrm{Aut}(X)$ contains a finite index normal subgroup $\mathrm{Aut}_0(X)$, the group of type-preserving automorphisms, which acts without inversions. Thus, any uniform lattice $\Gamma \leq \mathrm{Aut}(X)$ has a finite index subgroup $\Gamma \cap \mathrm{Aut}_0(X)$ which acts without inversions. Now, in Corollaries 2 and 3 below, the given sets of rational numbers are closed under multiplication by positive integers, so we need only consider uniform lattices which act without inversions. Note that the lattices described in Section 4.2 act without inversions, since they were constructed using complexes of groups.

The first result we will need is an analogue of Theorem 2. Let $F(X)$ be the set of faces, or 2-cells, of a polygonal complex X. Theorem 3 below gives a restriction on $F(X)$-covolumes.

THEOREM 3. *With the notation of Theorem 2, suppose Γ is a uniform lattice which acts without inversions. Then the $F(X)$-covolume of Γ is rational a/b (in lowest terms), such that b is not divisible by any primes other than $p_1, p_2, \ldots, p_M$. Moreover, if for some j we have $\alpha'_j = 0$, then b is not divisible by p_j.*

Note that no power of p_j can be a factor of b, in contrast to Theorem 2.

Proof. The $F(X)$-covolume is the finite sum

$$\mathrm{Vol}(\Gamma\backslash\backslash F(X)) = \sum_{\sigma \in \Gamma\backslash F(X)} \frac{1}{|\Gamma_\sigma|}$$

We claim that the order of each stabiliser Γ_σ is not divisible by any primes other that $p_1, p_2, \ldots, p_M$, and that if for some j we have $\alpha'_j = 0$, then the order of Γ_σ is not divisible by p_j. The proof of this claim is similar to that of the claim about orders of vertex stabilisers in Theorem 2, except that we begin the induction with the group of automorphisms without inversions of $B(\sigma, 0)$, which is trivial, so its order is not divisible by any p_j. □

We will also use the following consequence of Theorem 1. A similar result holds for tree lattices, and is used by Rosenberg to establish a restriction on the covolumes of uniform lattices acting on biregular trees in [**22**], Theorem 9.2.1.

COROLLARY 1. *Let X be a locally finite polygonal complex such that $G\backslash X$ is finite, where $G = \mathrm{Aut}(X)$. Then there is a constant $c(X)$, depending only on X,* *such that for all uniform lattices* Γ *which act without inversions,*

$$\mu(\Gamma\backslash G) = c(X)\,\mathrm{Vol}(\Gamma\backslash\backslash F(X))$$

Therefore, if both $\mu(\Gamma\backslash G)$ and $\mathrm{Vol}(\Gamma\backslash\backslash F(X))$ are known for just one uniform lattice Γ acting without inversions, the constant $c(X)$ may be computed. Using the examples of uniform lattices in Section 4.2, together with Theorem 3, we obtain the following results.

COROLLARY 2. *Let* X *be as in (1) of Section 4.1, and* $G = \mathrm{Aut}(X)$*. Then if* $\Gamma \leq G$ *is a uniform lattice,*

$$\mu(\Gamma\backslash G) = \frac{r}{mn}\left(\frac{a}{b}\right)$$

where a/b *is rational (in lowest terms), and the prime divisors of* b *are strictly less than* m*.*

In fact, we show in [**26**] that every rational number of the form given in Corollary 2 can be obtained as the covolume of some uniform lattice in G.

COROLLARY 3. *Let* X *be as in (2) of Section 4.1, and* $G = \mathrm{Aut}(X)$*. Then if* $\Gamma \leq G$ *is a uniform lattice,*

$$\mu(\Gamma\backslash G) = \frac{r}{[\mathcal{G} : B]}\left(\frac{a}{b}\right)$$

where B *is from the* BN*-pair of* $\mathcal{G}$*, and* a/b *is rational (in lowest terms), such that* b *is not divisible by any primes other than those dividing the order of a subgroup of* $\mathrm{Aut}_0(L)$ *which fixes an edge of* L *pointwise.*

The value of $[\mathcal{G} : B]$ can be computed from Table 1 of [**12**].

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