Covering theory for complexes of groups

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Abstract

We develop an explicit covering theory for complexes of groups, parallel to that developed for graphs of groups by Bass. Given a covering of developable complexes of groups, we construct the induced monomorphism of fundamental groups and isometry of universal covers. We characterize faithful complexes of groups and prove a conjugacy theorem for groups acting freely on polyhedral complexes. We also define an equivalence relation on coverings of complexes of groups, which allows us to construct a bijection between such equivalence classes, and subgroups or overgroups of a fixed lattice Γ in the automorphism group of a locally finite polyhedral complex X.

Key words: complexes of groups, covering theory, lattices

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1 Introduction

Let X be a locally finite polyhedral complex, such as a tree, a Davis complex, or a Bruhat–Tits building. Then the group G of automorphisms of X is naturally a locally compact group (see Section 2.2 below). Subgroups of G with prescribed properties, for example discreteness or cocompactness, may be encoded by graphs of groups (in the case of trees) or complexes of groups (for $\dim(X) \geq 2$). A covering theory for graphs of groups was developed by Bass in [1]. This has proved very useful for the study of tree lattices: see the reference Bass–Lubotzky [3].

The theory of complexes of groups is due to Gersten–Stallings [6], Corson [5] and Haefliger [7], [4]. Haefliger, in Chapter III.$\mathcal{C}$ of [4], translated into the framework of complexes of groups the general theory of coverings of étale groupoids. (Chapter III.$\mathcal{G}$ of [4] discusses groupoids of local isometries.) While covering theory for étale groupoids is powerful, as it is strictly parallel to the theory of coverings for topological spaces, the correspondence between coverings of complexes of groups and coverings of étale groupoids is not stated in [4]. Even the definition of the étale groupoid canonically associated to a complex of groups is quite involved (pp. 595–596 of [4]). The étale groupoid perspective thus does not easily yield results or constructions suitable for investigating concrete questions concerning group actions on polyhedral complexes.

One aim of this paper is to make covering theory for complexes of groups more accessible, by following a more explicit approach. We also prove several results for group actions which we hope will be broadly useful, including a characterization of faithful complexes of groups (in Section 3.3), and the Conjugacy Theorem (Theorem 3 below). Finally, we establish in Section 5 a bijection between suitably defined isomorphism classes of coverings and subgroups or overgroups of a fixed $\Gamma < \mathrm{Aut}(X)$. This forms the technical background for our work [9] on counting overlattices, and hopefully will have other applications.

Let us briefly recall Haefliger's theory of complexes of groups (see Section 2.4 below for details, and in particular for the definition of covering). The action of a group G on a simply connected polyhedral complex X induces a complex of groups $G(Y)$ over the quotient $Y = G\backslash X$. The fundamental group $\pi_1(G(Y))$ then acts on the simply connected universal cover $\widetilde{G(Y)}$, with $\pi_1(G(Y))$ isomorphic to G, and $\widetilde{G(Y)}$ isometric to X. An arbitrary complex of groups $G(Y)$ is developable if it is induced by a group action in this way. A key difference between Bass–Serre theory and the theory of complexes of groups is that complexes of groups need not be developable. However, if a complex of groups has nonpositive curvature (see Section 2.4.4), it is developable.

Our first main result describes the functoriality of coverings.

Theorem 1 *Let $\lambda : G(Y) \to G'(Y')$ be a covering of developable complexes of groups. Then λ induces a monomorphism of fundamental groups*

$$\Lambda : \pi_1(G(Y)) \to \pi_1(G'(Y'))$$

and a Λ-equivariant isometry of universal covers

$$L : \widetilde{G(Y)} \to \widetilde{G'(Y')}.$$

Theorem 1 also follows from covering theory for étale groupoids (Haefliger, private communication). Our contribution is to construct the maps Λ and L explicitly; we then make repeated use of these constructions in later sections of this work. Theorem 1 is proved in Section 3.2, using material from Section 3.1.

In Section 3.3, we characterize the group

$$N = \ker\left(\pi_1(G(Y)) \to \mathrm{Aut}(\widetilde{G(Y)})\right)$$

where $G(Y)$ is developable. If N is trivial, then the complex of groups $G(Y)$ is said to be *faithful*, and we may identify the fundamental group $\pi_1(G(Y))$ with a subgroup of $\mathrm{Aut}(\widetilde{G(Y)})$.

In Section 3.4 we develop technical results, similar to those in Section 4 of [1] in the case of trees. As described in Proposition 2.1 of [10], Haefliger's morphisms of complexes of groups, when restricted to complexes of groups over 1–dimensional spaces, are not the same as Bass' morphisms of graphs of groups. Also, the universal covers of graphs of groups and of complexes of groups are defined with respect to different choices. Hence, our proofs differ in many details from those of [1].

An additional consideration for complexes of groups, which has no analogue in Bass–Serre theory, is the relationship between coverings and developability. In Section 3.5, we show:

Proposition 2 *Let $\lambda : G(Y) \to G'(Y')$ be a covering of complexes of groups.*

(1) If $G'(Y')$ is developable, then $G(Y)$ is developable.
(2) If $G(Y)$ has nonpositive curvature (hence is developable), then $G'(Y')$ has nonpositive curvature, hence $G'(Y')$ is developable.

One of the main applications of the results of Section 3.4 is the Conjugacy Theorem below, proved as Theorem 47 in Section 4. Let H be a subgroup (acting without inversions) of $G = \mathrm{Aut}(X)$, for X a locally finite polyhedral complex, and define

$$G_H = \{g \in G \mid g\sigma \in H\sigma \text{ for all cells } \sigma \text{ of } X\}.$$

Theorem 3 (Conjugacy Theorem) *If $\Gamma \leq G_H$ acts freely on X then there is an element $g \in G_H$ such that $g\Gamma g^{-1} \leq H$.*

The corresponding result for trees (Theorem 5.2 of [1]) was a basic tool in [2]. In [1], as well as a proof using covering theory for graphs of groups, a simple direct proof due to the referee was given. This relied on the fact that a group acting freely on a tree is free. In higher dimensions, it seems that covering theory must be used.

In Section 5 we define isomorphism of coverings (see Definition 48) so that the following bijection holds:

Theorem 4 *Let X be a simply connected polyhedral complex, and let Γ be a subgroup of $\mathrm{Aut}(X)$ (acting without inversions) which induces a complex of groups $G(Y)$. Then there is a bijection between the set of subgroups of $\mathrm{Aut}(X)$ (acting without inversions) which contain Γ, and the set of isomorphism classes of coverings of faithful, developable complexes of groups by $G(Y)$.*

The main ingredients in the proof of Theorem 4 are Theorem 1 above, and the results of Section 3.4. As a corollary to Theorem 4, we show that there is a bijection between n–sheeted coverings, and overlattices of index n (that is, lattices containing a fixed lattice Γ with index n). Similar results hold for subgroups and sublattices. In [8], Lim defined isomorphism of coverings of graphs of groups and proved the bijection of Theorem 4 for trees.

2 Background

We begin by recalling the basic theory of lattices, in Section 2.1. Since the quotient of a simplicial complex by a simplicial group action is not in general a simplicial complex, it is natural to define complexes of groups over polyhedral complexes instead. In Section 2.2, we give definitions of polyhedral complexes and the topology of their automorphism groups. Small categories without loops, or scwols, are algebraic objects that substitute for polyhedral complexes. These are described in Section 2.3 (following section III.$\mathcal{C}$ 1-2 of [4]). We conclude this background material by, in Section 2.4, summarizing Haefliger's theory of complexes of groups, as presented in Chapter III.$\mathcal{C}$ of [4].

2.1 Lattices

Let G be a locally compact topological group with left-invariant Haar measure μ. A discrete subgroup Γ of G is a *lattice* if its covolume $\mu(\Gamma\backslash G)$ is finite. A lattice is called *cocompact* or *uniform* if $\Gamma\backslash G$ is compact.

Let $\mathcal{S}$ be a left G-set such that, for each $s \in \mathcal{S}$, the stabilizer G_s is compact and open. For any discrete subgroup Γ of G, the stabilizers Γ_s are finite groups, and we define the $\mathcal{S}$*-covolume* of Γ as

$$\mathrm{Vol}(\Gamma\backslash\backslash\mathcal{S}) = \sum_{s \in \Gamma\backslash\mathcal{S}} \frac{1}{|\Gamma_s|} \leq \infty.$$

It is shown in [3], Chapter 1, that if $G\backslash\mathcal{S}$ is finite and G admits a lattice, then there is a normalization of the Haar measure μ, depending only on $\mathcal{S}$, such that for every discrete subgroup Γ of G,

$$\mu(\Gamma\backslash G) = \mathrm{Vol}(\Gamma\backslash\backslash\mathcal{S}).$$

It is clear that for two lattices $\Gamma \subset \Gamma'$ of G, the index $[\Gamma' : \Gamma]$ is equal to the ratio of the covolumes $\mu(\Gamma\backslash G) : \mu(\Gamma'\backslash G)$.

2.2 Polyhedral complexes

Let M^n_κ be the complete, simply connected, Riemannian n-manifold of constant sectional curvature $\kappa \in \mathbb{R}$.

Definition 5 (polyhedral complex) *An M_κ-polyhedral complex K is a finite-dimensional CW-complex such that:*

(1) each open cell of dimension n is isometric to the interior of a compact convex polyhedron in M^n_κ; and
(2) for each cell σ of K, the restriction of the attaching map to each open codimension one face of σ is an isometry onto an open cell of K.

If an M_κ-polyhedral complex is locally finite, then it is a geodesic metric space by the Hopf–Rinow Theorem (see, for example, [4]). More generally, we have:

Theorem 6 (Bridson, [4]) *An M_κ-polyhedral complex with finitely many isometry classes of cells is a complete geodesic metric space.*

Let K be a locally finite, connected polyhedral complex, and let $\mathrm{Aut}(K)$ be the group of cellular isometries, or automorphisms, of K. Then $\mathrm{Aut}(K)$ is naturally a locally compact group, with a neighborhood basis of the identity consisting of automorphisms fixing larger and larger balls. With respect to this topology, a subgroup Γ of $\mathrm{Aut}(K)$ is discrete if and only if for each cell σ of K, the stabilizer Γ_σ is finite. A subgroup Γ of $\mathrm{Aut}(K)$ is said to act *without inversions* if whenever $g \in \Gamma$ preserves a cell of K, g fixes that cell pointwise.

In Chapter III.$\mathcal{C}$ of [4], complexes of groups are presented using the language of scwols, or small categories without loops. As we explain in this section, to any polyhedral complex K one may associate a scwol $\mathcal{X}$, which has a geometric realization $|\mathcal{X}|$ isometric to the barycentric subdivision of K. Morphisms of scwols correspond to polyhedral maps, and group actions on scwols correspond to actions without inversions on polyhedral complexes.

Definition 7 (scwol) *A small category without loops (scwol) $\mathcal{X}$ is a disjoint union of a set $V(\mathcal{X})$, the vertex set, and a set $E(\mathcal{X})$, the edge set, endowed with maps*

$$i : E(\mathcal{X}) \to V(\mathcal{X}) \quad \textit{and} \quad t : E(\mathcal{X}) \to V(\mathcal{X})$$

and, if $E^{(2)}(\mathcal{X})$ denotes the set of pairs (a, b) of edges where $i(a) = t(b)$, with a map

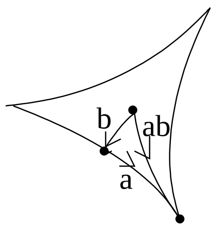
$$\begin{aligned} E^{(2)}(\mathcal{X}) &\to E(\mathcal{X}) \\ (a, b) &\mapsto ab \end{aligned}$$

such that:

(1) if $(a, b) \in E^{(2)}(\mathcal{X})$, then $i(ab) = i(b)$ and $t(ab) = t(a)$;
(2) if $a, b, c \in E(\mathcal{X})$ such that $i(a) = t(b)$ and $i(b) = t(c)$, then $(ab)c = a(bc)$; and
(3) if $a \in E(\mathcal{X})$, then $i(a) \neq t(a)$.

For $a \in E(\mathcal{X})$, the vertices $i(a)$ and $t(a)$ are called the *initial vertex* and *terminal vertex* of a respectively. If $(a, b) \in E^{(2)}(\mathcal{X})$ we say a and b are *composable*, and that ab is the *composition* of a and b. We will sometimes write $\alpha \in \mathcal{X}$ for $\alpha \in V(\mathcal{X}) \cup E(\mathcal{X})$. If $\alpha \in V(\mathcal{X})$ then $i(\alpha) = t(\alpha) = \alpha$.

The motivating example of a scwol is the scwol $\mathcal{X}$ associated to a polyhedral complex K. The set of vertices $V(\mathcal{X})$ corresponds to the set of cells of K (or the set of barycenters of the cells of K). The set of edges $E(\mathcal{X})$ is the set of 1-simplices of the barycentric subdivision of K, that is, each element of $E(\mathcal{X})$ corresponds to a pair of cells $T \subsetneq S$, with initial vertex S and terminal vertex T. The composition of the edge a corresponding to $T \subsetneq S$ and the edge b corresponding to $S \subsetneq U$ is the edge ab corresponding to $T \subsetneq U$.



Conversely, given a scwol $\mathcal{X}$, we may construct a polyhedral complex, called the *geometric realization*. For an integer $k \geq 0$, let $E^{(k)}(\mathcal{X})$ be the set of sequences $(a_1, a_2, \ldots, a_k)$ of composable edges, that is, $(a_j, a_{j+1}) \in E^{(2)}(\mathcal{X})$ if $k > 1$, $E^{(1)}(\mathcal{X}) = E(\mathcal{X})$, and $E^{(0)}(\mathcal{X}) = V(\mathcal{X})$. The geometric realization $|\mathcal{X}|$ of $\mathcal{X}$ is defined as a polyhedral complex whose cells of dimension k are standard k-simplices indexed by the elements of $E^{(k)}(\mathcal{X})$. For the details of this construction, see [4], pp. 522–523. If $\mathcal{X}$ is the scwol associated to an M_κ-polyhedral complex K, then $|\mathcal{X}|$ may be realized as an M_κ-polyhedral complex isometric to the barycentric subdivision of K.

For a scwol $\mathcal{X}$, let $E^{\pm}(\mathcal{X})$ be the set of *oriented edges*, that is, the set of symbols a^+ and a^-, where $a \in E(\mathcal{X})$. For $e = a^+$, we define $i(e) = t(a)$, $t(e) = i(a)$ and $e^{-1} = a^-$. For $e = a^-$, we define $i(e) = i(a)$, $t(e) = t(a)$ and $e^{-1} = a^+$.

An *edge path* in $\mathcal{X}$ joining the vertex σ to the vertex τ is a sequence $(e_1, e_2, \ldots, e_n)$ of elements of $E^{\pm}(\mathcal{X})$ such that $i(e_1) = \sigma$, $i(e_{j+1}) = t(e_j)$ for $1 \leq j \leq n-1$ and $t(e_n) = \tau$.

A scwol $\mathcal{X}$ is *connected* if for any two vertices $\sigma, \tau \in V(\mathcal{X})$, there is an edge path joining σ to τ. Equivalently, $\mathcal{X}$ is connected if and only if the geometric realization $|\mathcal{X}|$ is connected. A scwol is *simply connected* if and only if its geometric realization is simply connected as a topological space.

Definition 8 (morphism of scwols) *Let $\mathcal{X}$ and $\mathcal{X}'$ be two scwols. A morphism $l : \mathcal{X} \to \mathcal{X}'$ is a map that sends $V(\mathcal{X})$ to $V(\mathcal{X}')$ and $E(\mathcal{X})$ to $E(\mathcal{X}')$, such that*

(1) for each $a \in E(\mathcal{X})$, we have $i(l(a)) = l(i(a))$ and $t(l(a)) = l(t(a))$; and
(2) for each $(a, b) \in E^{(2)}(\mathcal{X})$, we have $l(ab) = l(a)l(b)$.
A nondegenerate morphism of scwols is a morphism of scwols such that in addition to (1) *and* (2),
(3) for each vertex $\sigma \in V(\mathcal{X})$, the restriction of l to the set of edges with initial vertex σ is a bijection onto the set of edges of $\mathcal{X}'$ with initial vertex $l(\sigma)$.

An *automorphism* of a scwol $\mathcal{X}$ is a morphism $l : \mathcal{X} \to \mathcal{X}$ which has an inverse. Note that Condition (3) in Definition 8 is automatic for automorphisms.

Definition 9 (covering of scwols) *Let $\mathcal{X}$ be a (nonempty) scwol and let $\mathcal{X}'$ be a connected scwol. A nondegenerate morphism of scwols $l : \mathcal{X} \to \mathcal{X}'$ is called a covering if, for every vertex σ of $\mathcal{X}$, the restriction of l to the set of edges with terminal vertex σ is a bijection onto the set of edges of $\mathcal{X}'$ with terminal vertex $l(\sigma)$.*

Let $\mathcal{X}$ and $\mathcal{X}'$ be scwols associated to polyhedral complexes K and K' respectively. A polyhedral map $K \to K'$ induces a morphism of scwols $\mathcal{X} \to \mathcal{X}'$, and conversely, a morphism $l : \mathcal{X} \to \mathcal{X}'$ induces a continuous polyhedral map $|l| : |\mathcal{X}| \to |\mathcal{X}'|$ (see [4], p. 526). The morphism l is nondegenerate if and only if the restriction of $|l|$ to the interior of each cell of K induces a homeomorphism onto the interior of a cell of K', and l is a covering if and only if $|l|$ is a (topological) covering. A morphism $l : \mathcal{X} \to \mathcal{X}$ is an automorphism of $\mathcal{X}$ if and only if $|l| : K \to K$ is an automorphism of K.

Definition 10 (group actions on scwols) *An action of a group G on a scwol $\mathcal{X}$ is a homomorphism from G to the group of automorphisms of $\mathcal{X}$ such that:*

(1) for all $a \in E(\mathcal{X})$ and $g \in G$, we have $g \cdot i(a) \neq t(a)$; and
(2) for all $g \in G$ and $a \in E(\mathcal{X})$, if $g \cdot i(a) = i(a)$ then $g \cdot a = a$ (no "inversions").

The action of a group G on a scwol $\mathcal{X}$ induces a *quotient scwol* $\mathcal{Y} = G\backslash\mathcal{X}$, defined as follows. The vertex set is $V(\mathcal{Y}) = G\backslash V(\mathcal{X})$ and the edge set $E(\mathcal{Y}) = G\backslash E(\mathcal{X})$. For every $a \in E(\mathcal{X})$ we have $i(Ga) = Gi(a)$ and $t(Ga) = Gt(a)$, and if $(a, b) \in E^{(2)}(\mathcal{X})$ then the composition of Ga and Gb is Gab. The natural projection $p : \mathcal{X} \to \mathcal{Y}$ is a nondegenerate morphism of scwols.

Let $\mathcal{X}$ be the scwol associated to a polyhedral complex K, and let Γ be a subgroup of $G = \mathrm{Aut}(K)$. Then Γ acts on $\mathcal{X}$, in the sense of Definition 10, if and only if Γ acts without inversions on K.

In the case K is locally finite, we define the covolume of a discrete subgroup $\Gamma \leq G$ acting on $\mathcal{X}$ as follows. For the Γ-set $\mathcal{S}$ in Section 2.1, we choose the set of vertices $V(\mathcal{X})$ (which corresponds to the set of cells of K). By the same arguments as for tree lattices ([3], Chapter 1), it can be shown that if $G\backslash K$ is finite, then Γ is a lattice if and only if its $V(\mathcal{X})$-covolume converges, and Γ is a cocompact lattice if and only if $\Gamma\backslash V(\mathcal{X})$ is a finite set. We now normalize the Haar measure μ on G so that

$$\mu(\Gamma\backslash G) = \mathrm{Vol}(\Gamma\backslash\backslash V(\mathcal{X})) = \sum_{\sigma \in \Gamma\backslash V(\mathcal{X})} \frac{1}{|\Gamma_\sigma|}.$$

2.4 Complexes of groups

In this section, we recall Haefliger's theory of complexes of groups. We mainly follow the notation and definitions of Chapter III.$\mathcal{C}$ of [4], although at times, such as in Defintion 17 and Proposition 27 below, we indicate choices and define maps more explicitly. Section 2.4.1 defines complexes of groups and their morphisms. Section 2.4.2 then discusses groups associated to complexes of groups, in particular the fundamental group, and Section 2.4.3 discusses scwols associated to complexes of groups, in particular the universal cover. In Section 2.4.4 we describe the role of local developments and nonpositive curvature. All references to [4] in this section are to Chapter III.$\mathcal{C}$, which the reader should consult for further details.

2.4.1 Objects and morphisms of the category of complexes of groups

Definition 11 (complex of groups) *Let $\mathcal{Y}$ be a scwol. A complex of groups $G(\mathcal{Y}) = (G_\sigma, \psi_a, g_{a,b})$ over $\mathcal{Y}$ is given by the following data:*

(1) for each $\sigma \in V(\mathcal{Y})$, a group G_σ, called the local group at σ;
(2) for each $a \in E(\mathcal{Y})$, an injective group homomorphism $\psi_a : G_{i(a)} \to G_{t(a)}$; and
(3) for each pair of composable edges $(a, b) \in E^{(2)}(\mathcal{Y})$, a twisting element $g_{a,b} \in G_{t(a)}$;

with the following properties:

(i) *$Ad(g_{a,b})\psi_{ab} = \psi_a\psi_b$, where $Ad(g_{a,b})$ denotes conjugation by $g_{a,b}$; and*
(ii) *$\psi_a(g_{b,c})g_{a,bc} = g_{a,b}g_{ab,c}$, for each triple $(a, b, c) \in E^{(3)}(\mathcal{Y})$.*

For example, any group G is a complex of groups over a singleton $\mathcal{Y} = \{*\} = V(\mathcal{Y})$, with $G_* = G$; since $E(\mathcal{Y}) = \phi$, no other data is necessary.

Definition 12 (morphism of complexes of groups) *Let $G(\mathcal{Y})$ be as in Definition 11 and let $G'(\mathcal{Y}') = (G'_{\sigma'}, \psi_{a'}, g_{a',b'})$ be another complex of groups over a scwol $\mathcal{Y}'$. Let $l : \mathcal{Y} \to \mathcal{Y}'$ be a morphism of scwols. A morphism $\phi = (\phi_\sigma, \phi(a)) : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ of complexes of groups over l consists of*

(1) a group homomorphism $\phi_\sigma : G_\sigma \to G'_{l(\sigma)}$, called the local map at σ, for each $\sigma \in V(\mathcal{Y})$; and
(2) an element $\phi(a) \in G'_{t(l(a))}$ for each $a \in E(\mathcal{Y})$;

such that:

(i) *$Ad(\phi(a))\psi_{l(a)}\phi_{i(a)} = \phi_{t(a)}\psi_a$; and*
(ii) *$\phi_{t(a)}(g_{a,b})\phi(ab) = \phi(a)\psi_{l(a)}(\phi(b))g_{l(a),l(b)}$, for every $(a, b) \in E^{(2)}(\mathcal{Y})$.*

A morphism ϕ is an *isomorphism* if l is an isomorphism of scwols and ϕ_σ is a group isomorphism for every $\sigma \in V(\mathcal{Y})$. A morphism ϕ is *injective on the local groups* if each of the maps ϕ_σ is injective.

The *composition* $\phi' \circ \phi$ of a morphism $\phi = (\phi_\sigma, \phi(a)) : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ over l and a morphism $\phi' = (\phi'_\sigma, \phi'(a)) : G'(\mathcal{Y}') \to G''(\mathcal{Y}'')$ over l' is the morphism over $l' \circ l$ defined by the homomorphisms $(\phi' \circ \phi)_\sigma = \phi'_{l(\sigma)} \circ \phi_\sigma$ and the elements $(\phi' \circ \phi)(a) = \phi'_{l(t(a))}(\phi(a))\phi'(l(a))$.

A special case of a morphism of complexes of groups is when $\mathcal{Y}'$ in Definition 12 is a singleton, with $G'_* = G'$. In this case, ϕ may be regarded as a morphism from the complex of groups $G(\mathcal{Y})$ to the group G'.

Definition 13 (homotopy) *Let ϕ and ϕ' be two morphisms from $G(\mathcal{Y})$ to a group G', given respectively by $(\phi_\sigma, \phi(a))$ and $(\phi'_\sigma, \phi'(a))$. A homotopy from ϕ to ϕ' is given by a family of elements $k_\sigma \in G'$, indexed by $\sigma \in V(\mathcal{Y})$, such that*

(1) $\phi'_\sigma = \mathrm{Ad}(k_\sigma)\phi_\sigma$ for all $\sigma \in V(\mathcal{Y})$; and
(2) $\phi'(a) = k_{t(a)}\phi(a)k_{i(a)}^{-1}$ for all $a \in E(\mathcal{Y})$.

Let G be a group acting on a scwol $\mathcal{X}$ with quotient $\mathcal{Y} = G\backslash\mathcal{X}$, and let $p : \mathcal{X} \to \mathcal{Y}$ be the natural projection. The *complex of groups* $G(\mathcal{Y}) = (G_\sigma, \psi_a, g_{a,b})$ *associated to the action of* G *on* $\mathcal{X}$ is defined as follows.

For each vertex $\sigma \in V(\mathcal{Y})$, choose a vertex $\overline{\sigma} \in V(\mathcal{X})$ such that $p(\overline{\sigma}) = \sigma$. For each edge $a \in E(\mathcal{Y})$ with $i(a) = \sigma$, there exists a unique edge $\overline{a} \in E(\mathcal{X})$ such that $p(\overline{a}) = a$ and $i(\overline{a}) = \overline{\sigma}$. Choose $h_a \in G$ such that $h_a \cdot t(\overline{a}) = \overline{t(a)}$. For each $\sigma \in V(\mathcal{Y})$, let G_σ be the stabilizer in G of $\overline{\sigma} \in V(\mathcal{X})$. For each $a \in E(\mathcal{Y})$, let $\psi_a : G_{i(a)} \to G_{t(a)}$ be conjugation by h_a, that is,

$$\psi_a : g \mapsto h_a g h_a^{-1}.$$

For every pair of composable edges $(a, b) \in E^{(2)}(\mathcal{Y})$, define $g_{a,b} = h_a h_b h_{ab}^{-1}$. Then $G(\mathcal{Y}) = (G_\sigma, \psi_a, g_{a,b})$ is a complex of groups.

When precision is needed, we denote the set of choices of $\overline{\sigma}$ and h_a in this construction by $C_\bullet$, and the complex of groups $G(\mathcal{Y})$ constructed with respect to these choices by $G(\mathcal{Y})_{C_\bullet}$. If $C'_\bullet$ is another choice of $\overline{\sigma}'$, h'_a, then an isomorphism $\phi = (\phi_\sigma, \phi(a))$ from $G(\mathcal{Y})_{C_\bullet}$ to $G(\mathcal{Y})_{C'_\bullet}$ is obtained by choosing elements $k_\sigma \in G$, such that for each $\sigma \in V(\mathcal{Y})$, $k_\sigma \cdot \overline{\sigma} = \overline{\sigma}'$. Then put $\phi_\sigma = \mathrm{Ad}(k_\sigma)|_{G_\sigma}$ and $\phi(a) = k_{t(a)} h_a k_{i(a)}^{-1} h_a'^{-1}$.

When $G(\mathcal{Y})$ is a complex of groups associated to an action of a group G, there is a canonical morphism of complexes of groups $\phi_1 : G(\mathcal{Y}) \to G$, given by $\phi_1 = (\phi_\sigma, \phi(a))$, with $\phi_\sigma = G_\sigma \to G$ the inclusion, and $\phi(a) = h_a$.

Definition 14 (developable) *A complex of groups $G(\mathcal{Y})$ is developable if it is isomorphic to a complex of groups associated to the action of a group G on a scwol $\mathcal{X}$ in the above sense, with $\mathcal{Y} = G\backslash\mathcal{X}$.*

Proposition 15 (Corollary 2.15, [4]) *A complex of groups $G(\mathcal{Y})$ is developable if and only if there exists a morphism ϕ from $G(\mathcal{Y})$ to some group G which is injective on the local groups.*

We now define coverings.

Definition 16 (covering of complexes of groups) *Let $\phi : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a morphism of complexes of groups over a nondegenerate morphism of scwols $l : \mathcal{Y} \to \mathcal{Y}'$, where $\mathcal{Y}'$ is connected. The morphism ϕ is a covering (of $G'(\mathcal{Y}')$ by $G(\mathcal{Y})$) if for each vertex $\sigma \in V(\mathcal{Y})$,*

(1) the group homomorphism $\phi_\sigma : G_\sigma \to G'_{l(\sigma)}$ is injective, and
(2) for every $a' \in E(\mathcal{Y}')$ and $\sigma \in V(\mathcal{Y})$ with $t(a') = \sigma' = l(\sigma)$, the map

$$\coprod_{\substack{a\in l^{-1}(a')\\ t(a)=\sigma}} G_\sigma/\psi_a(G_{i(a)}) \to G'_{\sigma'}/\psi_{a'}(G'_{i(a')})$$

induced by

$$g \mapsto \phi_\sigma(g)\phi(a)$$

is bijective.

From Condition (2) of this definition, it follows that

$$\sum_{\substack{a\in l^{-1}(a')\\ t(a)=\sigma}} \frac{|G_\sigma|}{|G_{i(a)}|} = \frac{|G'_{\sigma'}|}{|G'_{i(a')}|}.$$

Since $\mathcal{Y}'$ is connected, the value of

$$n := \sum_{\sigma\in\, l^{-1}(\sigma')} \frac{|G'_{\sigma'}|}{|G_\sigma|} = \sum_{a\in l^{-1}(a')} \frac{|G'_{i(a')}|}{|G_{i(a)}|}$$

is independent of the vertex σ' and the edge a'. A covering of complexes of groups with the above n is said to be *n–sheeted.*

We will often use Definition 17 below, which defines a morphism of complexes of groups induced by an equivariant morphism of scwols, keeping track of the choices we make.

Definition 17 (induced morphism) *Let $\mathcal{X}$ and $\mathcal{X}'$ be simply connected scwols, endowed with actions of groups G and G', and let $\mathcal{Y} = G\backslash\mathcal{X}$ and $\mathcal{Y}' = G'\backslash\mathcal{X}'$ be the quotient scwols. Let $L : \mathcal{X} \to \mathcal{X}'$ be a morphism of scwols which is equivariant with respect to a group morphism $\Lambda : G \to G'$. Let $l : \mathcal{Y} \to \mathcal{Y}'$ be the induced morphism of the quotients.*

For any choices $C_\bullet$ and $C'_\bullet$ of data for the actions of G and G' on $\mathcal{X}$ and $\mathcal{X}'$, and for any choice $N_\bullet$ of elements $k_\sigma \in G'$ indexed by $\sigma \in V(\mathcal{Y})$ such that $k_\sigma \cdot L(\overline{\sigma}) = \overline{l(\sigma)}$, there is an associated morphism of complexes of groups

$$\lambda = \lambda_{C_\bullet, C'_\bullet, N_\bullet} : G(\mathcal{Y})_{C_\bullet} \to G'(\mathcal{Y}')_{C'_\bullet}$$

over l, given by

$$\begin{aligned}\lambda_\sigma : G_\sigma &\to G'_{l(\sigma)}\\ g &\mapsto k_\sigma \Lambda(g) k_\sigma^{-1}\end{aligned}$$

and

$$\lambda(a) = k_{t(a)} \Lambda(h_a) k_{i(a)}^{-1} h'^{-1}_{l(a)}$$

(see Section 2.9(4), [4]).

2.4.2 The fundamental group of a complex of groups

There are two definitions of the fundamental group of a complex of groups, which result in canonically isomorphic groups. Both definitions involve the universal group.

Definition 18 (universal group) *The universal group $FG(\mathcal{Y})$ of a complex of groups $G(\mathcal{Y})$ over a scwol $\mathcal{Y}$ is the group presented by the generating set*

$$\coprod_{\sigma \in V(\mathcal{Y})} G_\sigma \coprod E^{\pm}(\mathcal{Y})$$

with the following relations:

(1) the relations in the groups G_σ;
(2) $(a^+)^{-1} = a^-$ and $(a^-)^{-1} = a^+$;
(3) $a^+ b^+ = g_{a,b}(ab)^+$, for every $(a, b) \in E^{(2)}(\mathcal{Y})$; and
(4) $\psi_a(g) = a^+ g a^-$, for every $g \in G_{i(a)}$.

There is a natural morphism $\iota = (\iota_\sigma, \iota(a)) : G(\mathcal{Y}) \to FG(\mathcal{Y})$, where $\iota_\sigma : G_\sigma \to FG(\mathcal{Y})$ takes G_σ to its image in $FG(\mathcal{Y})$, and $\iota(a) = a^+$.

Proposition 19 (Proposition 3.9, [4]) *A complex of groups $G(\mathcal{Y})$ over a connected scwol $\mathcal{Y}$ is developable if and only if $\iota : G(\mathcal{Y}) \to FG(\mathcal{Y})$ is injective on the local groups.*

The first definition of the fundamental group of a complex of groups $G(\mathcal{Y})$ involves the choice of a basepoint $\sigma_0 \in V(\mathcal{Y})$. A $G(\mathcal{Y})$*-path* starting from σ_0 is then a sequence $(g_0, e_1, g_1, e_2, \ldots, e_n, g_n)$ where $(e_1, e_2, \ldots, e_n)$ is an edge path in $\mathcal{Y}$ starting from σ_0, we have $g_0 \in G_{\sigma_0}$, and $g_j \in G_{t(e_j)}$ for $1 \le j \le n$. A $G(\mathcal{Y})$-path joining σ_0 to σ_0 is called a $G(\mathcal{Y})$*–loop* at σ_0.

To each path $c = (g_0, e_1, g_1, e_2, \ldots, e_n, g_n)$, we associate the element $\pi(c)$ of $FG(\mathcal{Y})$ represented by the word $g_0 e_1 g_1 \cdots e_n g_n$. Suppose now that c and $c' = (g'_0, e'_1, g'_1, \ldots, e'_n, g'_n)$ are two $G(\mathcal{Y})$–loops at σ_0. We say c and c' are *homotopic* if $\pi(c) = \pi(c')$, and denote the homotopy class of c by $[c]$. The *concatenation* *of c and c'* is the $G(\mathcal{Y})$–loop

$$c * c' = (g_0, e_1, \ldots, e_n, g_n g'_0, e'_1, \ldots, e'_{n'}, g'_{n'}).$$

The operation $[c][c'] = [c * c']$ defines a group structure on the set of homotopy classes of $G(\mathcal{Y})$–loops at σ_0.

Definition 20 (fundamental group of $G(\mathcal{Y})$ at σ_0) *The fundamental group of $G(\mathcal{Y})$ at σ_0 is the set of homotopy classes of $G(\mathcal{Y})$–loops at σ_0, with the group structure induced by concatenation. It is denoted $\pi_1(G(\mathcal{Y}), \sigma_0)$.*

Different choices of basepoint $\sigma_0 \in V(\mathcal{Y})$ result in isomorphic fundamental groups.

The second definition of the fundamental group of a complex of groups involves the choice of a maximal tree T in the 1-skeleton of the geometric realization $|\mathcal{Y}|$. By abuse of notation, we will say that T is a maximal tree in $\mathcal{Y}$.

Proposition 21 (Theorem 3.7, [4]) *For any maximal tree T in $\mathcal{Y}$, the fundamental group $\pi_1(G(\mathcal{Y}), \sigma_0)$ is isomorphic to the abstract group $\pi_1(G(\mathcal{Y}), T)$, presented by the generating set*

$$\coprod_{\sigma \in V(\mathcal{Y})} G_\sigma \coprod E^{\pm}(\mathcal{Y})$$

with the following relations:

(1) the relations in the groups G_σ;
(2) $(a^+)^{-1} = a^-$ and $(a^-)^{-1} = a^+$;
(3) $a^+ b^+ = g_{a,b}(ab)^+$, for every $(a, b) \in E^{(2)}(\mathcal{Y})$;
(4) $\psi_a(g) = a^+ g a^-$, for every $g \in G_{i(a)}$; and
(5) $a^+ = 1$ for every edge $a \in T$.

If $\mathcal{Y}$ is simply connected, then $\pi_1(G(\mathcal{Y}), T)$ is isomorphic to the direct limit of the diagram of groups G_σ and monomorphisms ψ_a. The isomorphism $\pi_1(G(\mathcal{Y}), \sigma_0) \to \pi_1(G(\mathcal{Y}), T)$ is the restriction of the natural projection $FG(\mathcal{Y}) \to \pi_1(G(\mathcal{Y}), T)$. Its inverse κ_T is defined in the proof of Proposition 27 in Section 2.4.3 below.

Let $\phi : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a morphism over a morphism of scwols $l : \mathcal{Y} \to \mathcal{Y}'$. Then ϕ induces a homomorphism $F\phi : FG(\mathcal{Y}) \to FG'(\mathcal{Y}')$, defined by $F\phi(g) = \phi_\sigma(g)$ for $g \in G_\sigma$, and $F\phi(a^+) = \phi(a) l(a)^+$. The restriction of $F\phi$ to $\pi_1(G(\mathcal{Y}), \sigma_0)$ is a natural homomorphism

$$\pi_1(\phi, \sigma_0) : \pi_1(G(\mathcal{Y}), \sigma_0) \to \pi_1(G'(\mathcal{Y}'), l(\sigma_0)).$$

In the particular case of a morphism $\phi : G(\mathcal{Y}) \to G$, where G is a group, the induced homomorphism

$$\pi_1(\phi, \sigma_0) : \pi_1(G(\mathcal{Y}), \sigma_0) \to G$$

is defined by $g \mapsto \phi_\sigma(g)$ for $g \in G_\sigma$, and $a^+ \mapsto \phi(a)$.

2.4.3 Developments and the universal cover

Any morphism from a complex of groups to a group induces a scwol, called the development.

Definition 22 (development) *Let $\phi : G(\mathcal{Y}) \to G$ be a morphism from a complex of groups $G(\mathcal{Y})$ to a group G. The scwol $D(\mathcal{Y}, \phi)$, called the development of $G(\mathcal{Y})$ with respect to ϕ, is defined as follows.*

The set of vertices is

$$V(D(\mathcal{Y}, \phi)) = \{([g], \sigma) : \sigma \in V(\mathcal{Y}), [g] \in G/\phi_\sigma(G_\sigma)\}$$

and the set of edges is

$$E(D(\mathcal{Y}, \phi)) = \{([g], a) : a \in E(\mathcal{Y}), [g] \in G/\phi_{i(a)}(G_{i(a)})\}.$$

The maps to initial and terminal vertices are given by

$$i([g], a) = ([g], i(a))$$

and

$$t([g], a) = ([g\phi(a)^{-1}], t(a))$$

and the composition of edges $([g], a)([h], b) = ([h], ab)$ is defined where $(a, b) \in E^{(2)}(\mathcal{Y})$, $g, h \in G$ and $g^{-1}h\phi(b)^{-1} \in \phi_{i(a)}(G_{i(a)})$.

The group G acts naturally on $D(\mathcal{Y}, \phi)$: given $g, h \in G$ and $\alpha \in \mathcal{Y}$, the action is $h \cdot ([g], \alpha) = ([hg], \alpha)$.

Proposition 23 (Theorems 2.13, 3.14 and 3.15, [4]) *Let $G(\mathcal{Y})$ be a complex of groups over a connected scwol $\mathcal{Y}$ and let G be a group.*

(1) Let $\phi : G(\mathcal{Y}) \to G$ be a morphism which is injective on the local groups. Then $G(\mathcal{Y})$ is the complex of groups (with respect to canonical choices) associated to the action of G on the development $D(\mathcal{Y}, \phi)$, and $\phi : G(\mathcal{Y}) \to G$ equals the canonical morphism $\phi_1 : G(\mathcal{Y}) \to G$.
(2) Suppose $G(\mathcal{Y})$ is a complex of groups associated to the action of G on a simply connected scwol $\mathcal{X}$, and $\phi_1 : G(\mathcal{Y}) \to G$ is the canonical morphism.

Then ϕ_1 induces a group isomorphism

$$\pi_1(\phi_1, \sigma_0) : \pi_1(G(\mathcal{Y}), \sigma_0) \xrightarrow{\sim} G$$

(see the final paragraph of Section 2.4.2), and there is a G-equivariant isomorphism of scwols

$$\Phi_1 : D(\mathcal{Y}, \phi_1) \xrightarrow{\sim} \mathcal{X}$$

given by, for $g \in G$ and $\alpha \in \mathcal{Y}$,

$$([g], \alpha) \mapsto g \cdot \overline{\alpha}.$$

The following result, on the functoriality of developments, is used to prove Theorem 1, stated in the introduction.

Proposition 24 (Theorem 2.18, [4]) *Let $G(\mathcal{Y})$ and $G'(\mathcal{Y}')$ be complexes of groups over scwols $\mathcal{Y}$ and $\mathcal{Y}'$. Let $\phi : G(\mathcal{Y}) \to G$ and $\phi' : G'(\mathcal{Y}') \to G'$ be morphisms to groups G and G' and let $\Lambda : G \to G'$ be a group homomorphism. Let $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a morphism over $l : \mathcal{Y} \to \mathcal{Y}'$.*

Suppose there is a homotopy from $\Lambda\phi$ to $\phi'\lambda$, given by elements $k_\sigma \in G'$ (see Definition 13). Then there is a Λ-equivariant morphism of the developments

$$L : D(\mathcal{Y}, \phi) \to D(\mathcal{Y}', \phi')$$

given by, for $g \in G$ and $\alpha \in \mathcal{Y}$,

$$([g], \alpha) \mapsto ([\Lambda(g) k_{i(\alpha)}^{-1}], l(\alpha)).$$

Moreover, if ϕ and ϕ' are injective on the local groups, and λ and Λ are isomorphisms, then L is an isomorphism of scwols.

We now define the universal cover.

Definition 25 (universal cover of a developable complex of groups) *Let $G(\mathcal{Y})$ be a developable complex of groups over a connected scwol $\mathcal{Y}$. Choose a maximal tree T in $\mathcal{Y}$. Let*

$$\iota_T : G(\mathcal{Y}) \to \pi_1(G(\mathcal{Y}), T)$$

be the morphism of complexes of groups mapping the local group G_σ to its image in $\pi_1(G(\mathcal{Y}), T)$, and the edge a to the image of a^+ in $\pi_1(G(\mathcal{Y}), T)$. The development $D(\mathcal{Y}, T) = D(\mathcal{Y}, \iota_T)$ is called a universal cover of $G(\mathcal{Y})$.

Theorem 26 (Theorem 3.13, [4]) *The universal cover $D(\mathcal{Y}, T)$ is connected and simply connected.*

As described in Definition 22, the fundamental group $\pi_1(G(\mathcal{Y}), T)$ acts canonically on $D(\mathcal{Y}, T)$.

A group action on a scwol induces the following explicit isomorphisms of groups and scwols.

Proposition 27 *Let G be a group acting on a simply connected scwol $\mathcal{X}$, and let $G(\mathcal{Y})$ be the induced complex of groups (with respect to some choices $C_\bullet = \{\overline{\sigma}, h_a\}$). Choose a maximal tree T in $\mathcal{Y}$ and a vertex $\sigma_0 \in V(\mathcal{Y})$. For $e \in E^{\pm}(\mathcal{Y})$, let*

$$h_e = \begin{cases} h_a & \text{if } e = a^+ \\ h_a^{-1} & \text{if } e = a^- \end{cases}.$$

For $\sigma \in V(\mathcal{Y})$, let $c_\sigma = (e_1, e_2, \ldots, e_n)$ be the unique edge-path contained in T, with no backtracking, which joins σ_0 to σ, and let $h_\sigma = h_{e_1} h_{e_2} \cdots h_{e_n}$.

Then there is a group isomorphism

$$\Lambda_T : \pi_1(G(\mathcal{Y}), T) \rightarrow G$$

defined on generators by

$$\begin{aligned} g &\mapsto h_\sigma g h_\sigma^{-1} \text{ for } g \in G_\sigma \\ a^+ &\mapsto h_{t(a)} h_a h_{i(a)}^{-1} \end{aligned}$$

and a Λ_T-equivariant isomorphism of scwols

$$\begin{aligned} \tilde{L}_T : D(\mathcal{Y}, T) &\rightarrow \mathcal{X} \\ ([g], \alpha) &\mapsto \Lambda_T(g) h_{i(\alpha)} \cdot \overline{\alpha}. \end{aligned}$$

PROOF. For $\sigma \in V(\mathcal{Y})$ let $\pi_\sigma = e_1 e_2 \cdots e_n$ be the element of $FG(\mathcal{Y})$ corresponding to the edge-path c_σ. Then by Theorem 3.7, [4], there is a canonical isomorphism

$$\kappa_T : \pi_1(G(\mathcal{Y}), T) \xrightarrow{\sim} \pi_1(G(\mathcal{Y}), \sigma_0)$$

defined on generators by

$$\begin{aligned} g &\mapsto \pi_\sigma g \pi_\sigma^{-1} \text{ for } g \in G_\sigma \\ a^+ &\mapsto \pi_{t(a)} a^+ \pi_{i(a)}^{-1}. \end{aligned}$$

By Proposition 23, the canonical morphism of complexes of groups $\phi_1 : G(\mathcal{Y}) \rightarrow G$ induces a group isomorphism $\pi_1(\phi_1, \sigma_0) : \pi_1(G(\mathcal{Y}), \sigma_0) \rightarrow G$. Composing κ_T with $\pi_1(\phi_1, \sigma_0)$, we obtain the group isomorphism $\Lambda_T : \pi_1(G(\mathcal{Y}), T) \xrightarrow{\sim} G$ defined above.

We now have the square

$$\begin{tikzcd} G(\mathcal{Y}) \arrow[r, "\iota_T"] \arrow[d, "\lambda=Id"'] & \pi_1(G(\mathcal{Y}),T) \arrow[d, "\Lambda_T"] \\ G(\mathcal{Y}) \arrow[r, "\phi_1"] & G. \end{tikzcd}$$

This commutes up to a homotopy from $\Lambda_T\iota_T$ to $\phi_1\lambda$, given by the elements h_σ^{-1}. Thus, by Proposition 24, there is a Λ_T-equivariant morphism of scwols

$$\begin{aligned} L_T : D(\mathcal{Y},T) &\to D(\mathcal{Y},\phi_1) \\ ([g],\alpha) &\mapsto ([\Lambda_T(g)h_{i(\alpha)}],\alpha) \end{aligned}$$

which is an isomorphism since ι_T and ϕ_1 are injective on the local groups, and both λ and Λ_T are isomorphisms. Composing L_T with the G-equivariant isomorphism $\Phi_1 : D(\mathcal{Y},\phi_1) \to \mathcal{X}$ (see Proposition 23), we obtain a Λ_T-equivariant isomorphism of scwols

$$\begin{aligned} \tilde{L}_T : D(\mathcal{Y},T) &\to \mathcal{X} \\ ([g],\alpha) &\mapsto \Lambda_T(g)h_{i(\alpha)} \cdot \overline{\alpha} \end{aligned}$$

as required.

2.4.4 Local developments and nonpositive curvature

Let K be a connected polyhedral complex and let $\mathcal{Y}$ be the scwol associated to K, so that $|\mathcal{Y}|$ is the first barycentric subdivision of K. The *star* $\mathrm{St}(\sigma)$ of a vertex $\sigma \in V(\mathcal{Y})$ is the union of the interiors of the simplices in $|\mathcal{Y}|$ which meet σ. If $G(\mathcal{Y})$ is a complex of groups over $\mathcal{Y}$, then each $\sigma \in V(\mathcal{Y})$ has a *local development*, even if $G(\mathcal{Y})$ is not developable. That is, we may naturally associate to each vertex $\sigma \in V(\mathcal{Y})$ an action of G_σ on some simplicial complex $\mathrm{St}(\tilde{\sigma})$ containing a vertex $\tilde{\sigma}$, such that $\mathrm{St}(\sigma)$ is the quotient of $\mathrm{St}(\tilde{\sigma})$ by the action of G_σ. If $G(\mathcal{Y})$ is developable, then for each $\sigma \in V(\mathcal{Y})$, the local development at σ is isomorphic to the star of each lift $\tilde{\sigma}$ of σ in the universal cover $D(\mathcal{Y},T)$.

We denote by $\mathrm{st}(\tilde{\sigma})$ the star of $\tilde{\sigma}$ in $\mathrm{St}(\tilde{\sigma})$.

Lemma 28 (Lemma 5.2, [4]) *Let $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a covering of complexes of groups, over a morphism of scwols $l : \mathcal{Y} \to \mathcal{Y}'$. Then for each $\sigma \in V(\mathcal{Y}')$, Condition (2) in the definition of a covering (Definition 16) is equivalent to the existence of a λ_σ-equivariant bijection* $\mathrm{st}(\tilde{\sigma}) \to \mathrm{st}(\widetilde{l(\sigma)})$.

In the case that $\mathcal{Y}$ is the scwol associated to a polyhedral complex K, each local development $\mathrm{St}(\tilde{\sigma})$ has a metric structure induced by that of K (see p. 562, [4]). A complex of groups $G(\mathcal{Y})$ has *nonpositive curvature* if for all

$\sigma \in V(\mathcal{Y})$, the local development at σ has nonpositive curvature (that is, $\mathrm{St}(\tilde{\sigma})$ is locally $\mathrm{CAT}(\kappa)$ for some $\kappa \leq 0$) in this induced metric. The importance of this condition is given by:

Theorem 29 (Theorem 4.17, [4]) *If a complex of groups has nonpositive curvature, it is developable.*

We will use the following condition to establish nonpositive curvature:

Lemma 30 (Remark 4.18, [4]) *Let $\mathcal{Y}$ be the scwol associated to an M_κ-polyhedral complex K, with $\kappa \leq 0$. Then $G(\mathcal{Y})$ has nonpositive curvature if and only if, for each vertex τ of K, the geometric link of $\tilde{\tau}$ in $\mathrm{st}(\tilde{\tau})$, with the induced spherical structure, is CAT(1).*

3 Covering theory for complexes of groups

This section contains our results for complexes of groups which are analogous to those for graphs of groups in [1]. We consider the functoriality of morphisms of complexes of groups in Section 3.1 and that of coverings in Section 3.2, culminating in the (constructive) proof of Theorem 1. Section 3.3 then characterizes faithfulness of complexes of groups. In Section 3.4 a key technical result, the Main Lemma (Lemma 40), is proved. The Main Lemma makes precise the relationship between maps of groups and scwols, and induced maps of fundamental groups and universal covers of complexes of groups. We consider the relationship between coverings and developability in Section 3.5; this has no analogy for graphs of groups since every graph of groups is developable.

3.1 Functoriality of morphisms

Proposition 31 below gives explicit constructions of the maps on fundamental groups and universal covers induced by a morphism of developable complexes of groups.

Proposition 31 *Let $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a morphism of complexes of groups over a morphism of scwols $l : \mathcal{Y} \to \mathcal{Y}'$, where $\mathcal{Y}$ and $\mathcal{Y}'$ are connected. Assume $G(\mathcal{Y})$ and $G'(\mathcal{Y}')$ are developable. For any choice of $\sigma_0 \in V(\mathcal{Y})$ and maximal trees T and T' in $\mathcal{Y}$ and $\mathcal{Y}'$ respectively, λ induces a homomorphism of fundamental groups*

$$\Lambda_{T,T'} = \Lambda^{\lambda}_{T,T'} : \pi_1(G(\mathcal{Y}), T) \to \pi_1(G'(\mathcal{Y}'), T')$$

and a $\Lambda_{T,T'}$-equivariant morphism of universal covers

$$L^{\lambda}_{T,T'} : D(\mathcal{Y}, T) \to D(\mathcal{Y}', T').$$

PROOF. Let $\sigma_0' = l(\sigma_0)$. Recall from the proof of Proposition 27 that there is a canonical isomorphism

$$\kappa_T : \pi_1(G(\mathcal{Y}), T) \xrightarrow{\sim} \pi_1(G(\mathcal{Y}), \sigma_0)$$

and from the last paragraph of Section 2.4.2 that the morphism λ induces a group homomorphism $\pi_1(\lambda, \sigma_0) : \pi_1(G(\mathcal{Y}), \sigma_0) \to \pi_1(G'(\mathcal{Y}'), \sigma_0')$ which is the restriction of the morphism $F\lambda : FG(\mathcal{Y}) \to FG'(\mathcal{Y}')$. The group homomorphism

$$\Lambda_{T,T'} : \pi_1(G(\mathcal{Y}), T) \to \pi_1(G'(\mathcal{Y}'), T')$$

is defined by the composition $\kappa'^{-1}_{T'} \circ \pi_1(\lambda, \sigma_0) \circ \kappa_T$:

$$\pi_1(G(\mathcal{Y}), T) \xrightarrow{\sim} \pi_1(G(\mathcal{Y}), \sigma_0) \longrightarrow \pi_1(G'(\mathcal{Y}'), \sigma_0') \xrightarrow{\sim} \pi_1(G'(\mathcal{Y}'), T')$$

We now have a square

$$\begin{tikzcd} G(\mathcal{Y}) \arrow[r, "\iota_T"] \arrow[d, "\lambda"'] & \pi_1(G(\mathcal{Y}), T) \arrow[d, "\Lambda_{T,T'}"'] \\ G'(\mathcal{Y}') \arrow[r, "\iota'_{T'}"] & \pi_1(G'(\mathcal{Y}'), T'). \end{tikzcd}$$

We claim that that there is a homotopy from $\Lambda_{T,T'} \circ \iota_T$ to $\iota'_{T'} \circ \lambda$. For $\sigma \in V(\mathcal{Y})$ let $\pi_\sigma = e_1 e_2 \cdots e_n$ be the element of $FG(\mathcal{Y})$ corresponding to the unique path $(e_1, e_2, \ldots, e_n)$ in T without backtracking from σ_0 to σ, and similarly for $\pi'_{l(\sigma)} \in FG'(\mathcal{Y}')$. Then for $g \in G_\sigma$, we have

$$\begin{aligned}
(\Lambda_{T,T'} \circ \iota_T)(g) =& \Lambda_{T,T'}(g) \\
=& \kappa'^{-1}_{T'} \circ \pi_1(\lambda, \sigma_0) \circ \kappa_T(g) \\
=& \kappa'^{-1}_{T'} \circ \pi_1(\lambda, \sigma_0)(\pi_\sigma g \pi_\sigma^{-1}) \\
=& \kappa'^{-1}_{T'} \{F\lambda(\pi_\sigma) \lambda_\sigma(g) (F\lambda(\pi_\sigma))^{-1}\} \\
=& \kappa'^{-1}_{T'} \{F\lambda(\pi_\sigma)(\pi'_{l(\sigma)})^{-1}\} \, (\iota'_{T'} \circ \lambda_\sigma)(g) \, \kappa'^{-1}_{T'} \{\pi'_{l(\sigma)} (F\lambda(\pi_\sigma))^{-1}\}.
\end{aligned}$$

Setting

$$u_\sigma = \kappa'^{-1}_{T'} \{F\lambda(\pi_\sigma)(\pi'_{l(\sigma)})^{-1}\} \in \pi_1(G'(\mathcal{Y}'), T')$$

we conclude

$$(\Lambda_{T,T'} \circ \iota_T)(g) = u_\sigma (\iota'_{T'} \circ \lambda_\sigma)(g) \, u_\sigma^{-1} = \mathrm{Ad}(u_\sigma)(\iota'_{T'} \circ \lambda)(g).$$

Similarly, if $a \in E(\mathcal{Y})$, we compute

$$\begin{aligned}
(\Lambda_{T,T'} \circ \iota_T)(a) &= \kappa'^{-1}_{T'} \circ \pi_1(\lambda, \sigma_0) \circ \kappa_T(a^+) \\
&= \kappa'^{-1}_{T'} \circ \pi_1(\lambda, \sigma_0)(\pi_{t(a)} a^+ \pi^{-1}_{i(a)}) \\
&= u_{t(a)} \lambda(a) l(a)^+ u^{-1}_{i(a)} \\
&= u_{t(a)} (\iota'_{T'} \circ \lambda)(a) u^{-1}_{i(a)}.
\end{aligned}$$

The last equality comes from the definition of composition of morphisms,

$$(\iota'_{T'} \circ \lambda)(a) = (\iota'_{T'})_{l(t(a))}(\lambda(a))\iota'_{T'}(l(a)) = \lambda(a) l(a)^+.$$

Hence the desired homotopy from $\Lambda_{T,T'} \circ \iota_T$ to $\iota'_{T'} \circ \lambda$ is given by the elements u_σ^{-1}.

By Proposition 24 there is thus a $\Lambda_{T,T'}$-equivariant morphism of universal covers

$$L^\lambda_{T,T'} : D(\mathcal{Y}, T) \to D(\mathcal{Y}', T')$$

given by

$$([g], \alpha) \mapsto ([\Lambda_{T,T'}(g) u_{i(\alpha)}], l(\alpha)).$$

Corollary 32 below says that if a diagram of morphisms of developable complexes of groups commutes, then the corresponding diagrams of the induced maps on fundamental groups and universal covers, defined in Proposition 31 above, also commute.

Corollary 32 *With the notation of Proposition 31, let $G''(\mathcal{Y}'')$ be a developable complex of groups over a connected scwol $\mathcal{Y}''$, and assume there is a morphism $\lambda' : G'(\mathcal{Y}') \to G''(\mathcal{Y}'')$. Choose a maximal tree T'' in $\mathcal{Y}''$. Then the composition*

$$\lambda'' = \lambda' \circ \lambda$$

induces a group homomorphism $\Lambda_{T,T''} : \pi_1(G(\mathcal{Y}), T) \to \pi_1(G''(\mathcal{Y}''), T'')$ and a $\Lambda_{T,T''}$-equivariant morphism of universal covers $L^{\lambda''}_{T,T''} : D(\mathcal{Y}, T) \to D(\mathcal{Y}'', T'')$, such that

$$L^{\lambda''}_{T,T''} = L^{\lambda'}_{T',T''} \circ L^\lambda_{T,T'}$$

and

$$\Lambda_{T,T''} = \Lambda_{T',T''} \circ \Lambda_{T,T'}.$$

PROOF. The proof follows from the constructions given in Proposition 31 above, and the definition of composition of morphisms.

In this section we prove Theorem 1, stated in the Introduction. The maps $\Lambda_{T,T'}$ and $L^{\lambda}_{T,T'}$ are those defined in Proposition 31 above.

Proposition 33 *Let $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a covering of complexes of groups over a morphism of scwols $l : \mathcal{Y} \to \mathcal{Y}'$, where $\mathcal{Y}$ and $\mathcal{Y}'$ are connected. Assume $G(\mathcal{Y})$ and $G'(\mathcal{Y}')$ are developable. For any choice of $\sigma_0 \in V(\mathcal{Y})$ and maximal trees T and T' in $\mathcal{Y}$ and $\mathcal{Y}'$ respectively, the induced homomorphism of fundamental groups*

$$\Lambda_{T,T'} : \pi_1(G(\mathcal{Y}), T) \to \pi_1(G'(\mathcal{Y}'), T')$$

is a monomorphism and

$$L^{\lambda}_{T,T'} : D(\mathcal{Y}, T) \to D(\mathcal{Y}', T')$$

is a $\Lambda_{T,T'}$-equivariant isomorphism of scwols.

PROOF. We begin with Lemma 34 below, which shows that $L^{\lambda}_{T,T'}$ is a covering of scwols (see Definition 9). Corollary 35 of this lemma shows that $L^{\lambda}_{T,T'}$ is an isomorphism of scwols. We then use this result to show that $\Lambda_{T,T'}$ is injective.

Lemma 34 *The morphism $L^{\lambda}_{T,T'}$ is a covering of scwols.*

PROOF. Let $g \in \pi_1(G(\mathcal{Y}), T)$ and $\sigma \in V(\mathcal{Y})$.

We first show that $L^{\lambda}_{T,T'}$ is injective on the set of edges with terminal vertex $([g], \sigma)$. Suppose a_1 and a_2 are edges of $\mathcal{Y}$ (with $t(a_1) = t(a_2) = \sigma$), that for some $h_1, h_2 \in \pi_1(G(\mathcal{Y}), T)$

$$t\left([h_1], a_1\right) = ([g], \sigma) = t\left([h_2], a_2\right)$$

and that

$$L^{\lambda}_{T,T'}\left([h_1], a_1\right) = L^{\lambda}_{T,T'}\left([h_2], a_2\right).$$

By definition of $L^{\lambda}_{T,T'}$, we then have $l(a_1) = l(a_2) = a'$ say, with $t(a') = l(t(a_1)) = l(\sigma) = \sigma'$. Also, by definition of the map $t : E(D(\mathcal{Y}, T)) \to V(D(\mathcal{Y}, T))$, we have, for some $h \in G_\sigma$,

$$h_1 a_1^- = h_2 a_2^- h^{-1}.$$

Now by definition of $L^\lambda_{T,T'}$, it follows that the group $G'_{i(a')}$ contains

$$\begin{aligned}
&\left(\Lambda_{T,T'}(h_1)u_{i(a_1)}\right)^{-1}\left(\Lambda_{T,T'}(h_2)u_{i(a_2)}\right)\\
&= u^{-1}_{i(a_1)}\Lambda_{T,T'}\left(a_1^-\,h\,a_2^+\right)u_{i(a_2)}\\
&= u^{-1}_{i(a_1)}u_{i(a_1)}l(a_1)^-\lambda(a_1)^{-1}u^{-1}_{t(a_1)}u_\sigma\lambda_\sigma(h)u_\sigma^{-1}u_{t(a_2)}\lambda(a_2)l(a_2)^+u^{-1}_{i(a_2)}u_{i(a_2)}\\
&= a'^-\lambda(a_1)^{-1}\lambda_\sigma(h)\lambda(a_2)a'^+.
\end{aligned}$$

Thus by the relation $a'^+ka'^- = \psi_{a'}(k)$, for all $k \in G'_{i(a')}$,

$$\lambda(a_1)^{-1}\,\lambda_\sigma(h)\,\lambda(a_2) \in \psi_{a'}(G'_{i(a')}).$$

That is, $\lambda(a_1)$ and $\lambda_\sigma(h)\lambda(a_2)$ belong to the same coset of $\psi_{a'}(G'_{i(a')})$ in $G'_{\sigma'}$. By Condition (2) in the definition of a covering (Definition 16, this implies $a_1 = a_2 = a$, say, and $h \in \psi_a(G_{i(a)})$. It follows that h_1 and h_2 belong to the same coset of $G_{i(a)}$ in $\pi_1(G(\mathcal{Y}),T)$. Thus $L^\lambda_{T,T'}$ is injective on the set of edges with terminal vertex $([g],\sigma)$.

We now show that $L^\lambda_{T,T'}$ surjects onto the set of edges of $D(\mathcal{Y}',T')$ with terminal vertex $L^\lambda_{T,T'}([g],\sigma)$. Suppose

$$t\left([h'],a'\right) = L^\lambda_{T,T'}\left([g],\sigma\right)$$

where $h' \in \pi_1(G'(\mathcal{Y}'),T')$, $a' \in E(\mathcal{Y}')$. Then $t(a') = \sigma' = l(\sigma)$ and by definition of $L^\lambda_{T,T'}$,

$$h'a'^- = \Lambda_{T,T'}(g)u_\sigma k_{\sigma'} \tag{1}$$

for some $k_{\sigma'} \in G'_{\sigma'}$. By Condition 2 in the definition of a covering, there exists an edge $a \in E(\mathcal{Y})$ with $l(a) = a'$ and $t(a) = \sigma$, and an element $k_\sigma \in G_\sigma$, such that $\lambda_\sigma(k_\sigma)\lambda(a)$ and $k_{\sigma'}$ belong to the same coset of $\psi_{a'}(G_{i(a')})$ in $G'_{\sigma'}$. Let $h = gk_\sigma a^+ \in \pi_1(G(\mathcal{Y}),T)$ and note that by Definition 22,

$$t\left([h],a\right) = \left([gk_\sigma a^+\iota_T(a)^{-1}],t(a)\right) = \left([gk_\sigma a^+a^-],\sigma\right) = \left([gk_\sigma],\sigma\right) = \left([g],\sigma\right).$$

We claim

$$L^\lambda_{T,T'}\left([h],a\right) = \left([h'],a'\right).$$

By Equation (1) above, the choice of a and k_σ and the relation $\psi_{a'}(k') = a'^+k'a'^-$ for all $k' \in G'_{i(a')}$, we have

$$\begin{aligned}
\Lambda_{T,T'}(h)u_{i(a)} &= \Lambda_{T,T'}(g)u_\sigma\lambda_\sigma(k_\sigma)u_\sigma^{-1}u_{t(a)}\lambda(a)l(a)^+u^{-1}_{i(a)}u_{i(a)}\\
&= h'a'^-k^{-1}_{\sigma'}\lambda_\sigma(k_\sigma)\lambda(a)a'^+\\
&\in h'G'_{i(a')}.
\end{aligned}$$

Hence,

$$L^\lambda_{T,T'}\left([h],a\right) = \left(\left[\Lambda_{T,T'}(h)u_{i(a)}\right],a'\right) = ([h'],a').$$

We conclude that $L^\lambda_{T,T'}$ is a covering of scwols.

Corollary 35 *Under the assumptions of Proposition 33, the morphism* $L^{\lambda}_{T,T'} : D(\mathcal{Y}, T) \to D(\mathcal{Y}', T')$ *is an isomorphism of scwols.*

PROOF. By Lemma 34, $L^{\lambda}_{T,T'}$ is a covering morphism. Since $D(\mathcal{Y}', T')$ is connected, $L^{\lambda}_{T,T'}$ is surjective, and since $D(\mathcal{Y}, T)$ is connected and $D(\mathcal{Y}', T')$ is simply connected, $L^{\lambda}_{T,T'}$ is injective. See Remark 1.9(2), [4].

We complete the proof of Proposition 33 by showing that $\Lambda_{T,T'}$ is a monomorphism of groups. Suppose $g \in \pi_1(G(\mathcal{Y}), T)$ and $\Lambda_{T,T'}(g) = 1$. Since $L^{\lambda}_{T,T'}$ is injective and $\Lambda_{T,T'}$-equivariant, g must act trivially on $D(\mathcal{Y}, T)$. In particular,

$$g \cdot ([1], \sigma_0) = ([g], \sigma_0) = ([1], \sigma_0)$$

so $g \in G_{\sigma_0}$. We then calculate

$$\begin{aligned}\Lambda_{T,T'}(g) &= \kappa'^{-1}_{T'} \circ \pi_1(\lambda, \sigma_0) \circ \kappa_T((\iota_T)_{\sigma_0}(g)) \\ &= \kappa'^{-1}_{T'}(\lambda_{\sigma_0}((\iota_T)_{\sigma_0}(g))) \\ &= 1.\end{aligned}$$

Since $\kappa'^{-1}_{T'}$, λ_{σ_0} and $(\iota_T)_{\sigma_0}$ are each injective, this implies $g = 1$. Thus $\Lambda_{T,T'}$ is injective.

Corollary 36 *Let* $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ *be a covering of complexes of groups. Suppose for some* $\kappa \in \mathbb{R}$ *that the scwols* $\mathcal{Y}$ *and* $\mathcal{Y}'$ *are associated to* M_κ*-polyhedral complexes with finitely many isometry classes of cells. If* $G(\mathcal{Y})$ *and* $G'(\mathcal{Y}')$ *are developable, then the geometric realizations of their respective universal covers are isometric (as polyhedral complexes).*

3.3 Faithfulness

Definition 37 (faithful) *Let* $G(\mathcal{Y})$ *be a developable complex of groups. We say* $G(\mathcal{Y})$ *is faithful if the natural homomorphism* $\pi_1(G(\mathcal{Y}), T) \to \mathrm{Aut}(D(\mathcal{Y}, T))$ *is a monomorphism, for any choice of maximal tree* T *in* $\mathcal{Y}$*.*

If $G(\mathcal{Y})$ is a complex of groups associated to the action of a group G on a scwol $\mathcal{X}$, then $G(\mathcal{Y})$ is faithful.

Proposition 38 below may be used to give sufficient conditions for faithfulness.

Proposition 38 *Let* $G(\mathcal{Y})$ *be a developable complex of groups over a connected scwol* $\mathcal{Y}$*. Choose a maximal tree* T *in* $\mathcal{Y}$*, and identify each local group*

G_σ with its image in $\pi_1(G(\mathcal{Y}),T)$ under the morphism ι_T. Let

$$N_T = \ker(\pi_1(G(\mathcal{Y}),T) \to D(\mathcal{Y},T)).$$

Then

(1) N_T is a vertex subgroup, that is $N_T \leq G_\sigma$ for each $\sigma \in V(\mathcal{Y})$.
(2) N_T is $\mathcal{Y}$-invariant, that is $\psi_a(N_T) = N_T$ for each $a \in E(\mathcal{Y})$.
(3) N_T is normal, that is $N_T \trianglelefteq G_\sigma$ for each $\sigma \in V(\mathcal{Y})$.
(4) N_T is maximal: if N'_T is another $\mathcal{Y}$-invariant normal vertex subgroup then $N'_T \leq N_T$.

PROOF. If $h \in N_T$, then for all $\sigma \in V(\mathcal{Y})$,

$$h \cdot ([1],\sigma) = ([h],\sigma) = ([1],\sigma)$$

thus $h \in G_\sigma$. This proves (1). Since N_T is normal in $\pi_1(G(\mathcal{Y}),T)$ it is normal in each G_σ, proving (3).

To prove (2), let $a \in E(\mathcal{Y})$. In the group $\pi_1(G(\mathcal{Y}),T)$ the following relation holds for each $g \in G_{i(a)}$:

$$\psi_a(g) = a^+ g a^-.$$

Since N_T is a subgroup of $G_{i(a)}$ and N_T is normal in $\pi_1(G(\mathcal{Y}),T)$, it follows that

$$\psi_a(N_T) = a^+ N_T a^- = N_T$$

as required.

To prove (4), we have, for all $g \in \pi_1(G(\mathcal{Y}),T)$ and $\alpha \in \mathcal{Y}$,

$$N'_T \cdot ([g],\alpha) = gN'_T g^{-1} \cdot ([g],\alpha) = g \cdot ([1],\alpha) = ([g],\alpha)$$

since N'_T is normal in $\pi_1(G(\mathcal{Y}),T)$ and N'_T is a subgroup of $G_{i(\alpha)}$. Hence N'_T is contained in N_T, as claimed.

3.4 Other functoriality results

This section contains results similar to those in Section 4, [1].

We first prove the following useful characterization of isomorphisms of complexes of groups. This result corresponds to Corollary 4.6, [1].

Proposition 39 *Let $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a morphism of developable complexes of groups over a morphism of scwols $l : \mathcal{Y} \to \mathcal{Y}'$, where $\mathcal{Y}$ and $\mathcal{Y}'$ are connected scwols. For any choice of $\sigma_0 \in V(\mathcal{Y})$ and maximal trees T and T'*

in $\mathcal{Y}$ and $\mathcal{Y}'$ respectively, λ is an isomorphism if and only if both of the maps $L^{\lambda}_{T,T'}$ and $\Lambda_{T,T'}$ are isomorphisms.

PROOF. If λ is an isomorphism, it is clearly a covering. Proposition 33 thus implies that $L^{\lambda}_{T,T'}$ is an isomorphism of scwols and $\Lambda_{T,T'}$ is a monomorphism of groups. Since λ^{-1} is also a covering, $\Lambda^{-1}_{T,T'} = (\Lambda_{T,T'})^{-1}$ is also a monomorphism, hence $\Lambda_{T,T'}$ is an isomorphism.

Conversely, suppose λ is not an isomorphism, thus one of λ and λ^{-1} is not a covering. Without loss of generality, we assume λ is not a covering. Then either

(1) there is a homomorphism $\lambda_\sigma : G_\sigma \to G'_{l(\sigma)}$ which is not injective, or
(2) there exists $a' \in E(\mathcal{Y}')$ and $\sigma \in V(\mathcal{Y})$ with $t(a') = \sigma' = l(\sigma)$, such that the map

$$\coprod_{\substack{a\in l^{-1}(a')\\ t(a)=\sigma}} G_\sigma/\psi_a(G_{i(a)}) \to G'_{\sigma'}/\psi_{a'}(G'_{i(a')})$$

induced by

$$g \mapsto \lambda_\sigma(g)\lambda(a)$$

is not bijective.

Condition (1) implies that the map $\Lambda_{T,T'}$ is not a monomorphism at G_σ, thus $\Lambda_{T,T'}$ is not an isomorphism. Condition (2) implies that $L^{\lambda}_{T,T'}$ is not a local bijection at $\mathrm{St}(\tilde{\sigma})$ (see Remark 5.3, [4]), thus the map $L^{\lambda}_{T,T'}$ is not an isomorphism.

The Main Lemma below, which corresponds to Proposition 4.4, [1], will be used many times in Section 5. The data for the Main Lemma is as follows.

Let $\mathcal{X}$ and $\mathcal{X}'$ be simply connected scwols, acted upon by groups G and G' respectively, with quotient scwols $\mathcal{Y} = G\backslash\mathcal{X}$ and $\mathcal{Y}' = G'\backslash\mathcal{X}'$. Let $G(\mathcal{Y})_{C_\bullet}$ and $G'(\mathcal{Y}')_{C'_\bullet}$ be complexes of groups associated to the actions of G and G', with respect to choices $C_\bullet = (\overline{\sigma}, h_a)$ and $C'_\bullet = (\overline{\sigma'}, h_{a'})$.

Suppose $L : \mathcal{X} \to \mathcal{X}'$ is a morphism of scwols which is equivariant with respect to some group homomorphism $\Lambda : G \to G'$. Let $l : \mathcal{Y} \to \mathcal{Y}'$ be the induced morphism of quotient scwols. Fix $\sigma_0 \in \mathcal{Y}$ and let $\sigma'_0 = l(\sigma_0)$. Let $N_\bullet = \{k_\sigma\}$ be a set of elements of G' such that $k_\sigma \cdot L(\overline{\sigma}) = \overline{l(\sigma)}$ for all $\sigma \in V(\mathcal{Y})$.

With respect to these choices, there is an induced morphism $\lambda = \lambda_{C_\bullet, C'_\bullet, N_\bullet}$: $G(\mathcal{Y}) \to G'(\mathcal{Y}')$ (see Definition 17). For any choice of maximal trees T and T' in $\mathcal{Y}$ and $\mathcal{Y}'$, respectively, let

$$\Lambda^{\lambda}_{T,T'} : \pi_1(G(\mathcal{Y}), T) \to \pi_1(G'(\mathcal{Y}'), T')$$

be the homomorphism of groups induced by λ and let

$$L^{\lambda}_{T,T'} : D(\mathcal{Y}, T) \to D(\mathcal{Y}', T')$$

be the associated $\Lambda^{\lambda}_{T,T'}$-equivariant morphism of scwols (see Proposition 31). By Proposition 27 we have isomorphisms of scwols

$$\tilde{L}_T : D(\mathcal{Y}, T) \xrightarrow{\sim} \mathcal{X} \quad \text{and} \quad \tilde{L}_{T'} : D(\mathcal{Y}', T') \xrightarrow{\sim} \mathcal{X}'$$

which are equivariant with respect to group isomorphisms

$$\Lambda_T : \pi_1(G(\mathcal{Y}), T) \xrightarrow{\sim} G \quad \text{and} \quad \Lambda_{T'} : \pi_1(G'(\mathcal{Y}'), T') \xrightarrow{\sim} G'$$

respectively.

Lemma 40 (Main Lemma) *Suppose $C_\bullet$ and $C'_\bullet$ are chosen so that $L(\overline{\sigma_0}) = \overline{l(\sigma_0)} = \overline{\sigma'_0}$, and $N_\bullet$ is chosen so that $k_{\sigma_0} = 1$. Then the following diagrams commute:*

(1)

$$\begin{tikzcd}
\pi_1(G(\mathcal{Y}), T) \arrow[r, "\Lambda^{\lambda}_{T,T'}"] \arrow[d, "\Lambda_T"] & \pi_1(G'(\mathcal{Y}'), T') \arrow[d, "\Lambda_{T'}"] \\
G \arrow[r, "\Lambda"] & G'
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
D(\mathcal{Y}, T) \arrow[r, "L^{\lambda}_{T,T'}"] \arrow[d, "\tilde{L}_T"] & D(\mathcal{Y}', T') \arrow[d, "\tilde{L}_{T'}"] \\
\mathcal{X} \arrow[r, "L"] & \mathcal{X}'.
\end{tikzcd}$$

PROOF. We first show the commutativity of (1), and then use this diagram and equivariance to prove that (2) commutes.

By construction,

$$\Lambda_T = \pi_1(\phi_1, \sigma_0) \circ \kappa_T \quad \text{and} \quad \Lambda_{T'} = \pi_1(\phi'_1, \sigma'_0) \circ \kappa'_{T'}$$

where $\phi_1 : G(\mathcal{Y}) \to G$ and $\phi'_1 : G'(\mathcal{Y}') \to G'$ are the canonical morphisms. Also, $\Lambda^{\lambda}_{T,T'} = \kappa'^{-1}_{T'} \circ \pi_1(\lambda, \sigma_0) \circ \kappa_T$. Therefore it is enough to show that the following diagram commutes:

$$\begin{tikzcd}
\pi_1(G(\mathcal{Y}), \sigma_0) \arrow[r, "\pi_1(\lambda,\sigma_0)"] \arrow[d, "\pi_1(\phi_1,\sigma_0)"] & \pi_1(G'(\mathcal{Y}'), \sigma'_0) \arrow[d, "\pi_1(\phi'_1,\sigma'_0)"] \\
G \arrow[r, "\Lambda"] & G'.
\end{tikzcd}$$

Let $x \in \pi_1(G(\mathcal{Y}), \sigma_0)$. Then x has the form

$$x = g_{\sigma_0} e_1 g_{\sigma_1} \cdots e_n g_{\sigma_n}$$

where $(g_{\sigma_0}, e_1, g_{\sigma_1}, \ldots, e_n, g_{\sigma_n})$ is a $G(\mathcal{Y})$–loop based at $\sigma_0 = \sigma_n$. It follows that

$$\pi_1(\phi_1, \sigma_0)(x) = g_{\sigma_0} h_{e_1} g_{\sigma_1} \cdots h_{e_n} g_{\sigma_n}$$

where the elements h_{e_j} are as defined in Proposition 27. We now compute

$$\begin{aligned}
&\pi_1(\phi'_1, \sigma'_0) \circ \pi_1(\lambda, \sigma_0)(x) \\
&\quad = (k_{\sigma_0}\Lambda(g_{\sigma_0})k_{\sigma_0}^{-1})(k_{\sigma_0}\Lambda(h_{e_1})k_{\sigma_1}^{-1}h_{l(e_1)}^{-1})h_{l(e_1)}(k_{\sigma_1}\Lambda(g_{\sigma_1})k_{\sigma_1}^{-1})\cdots(k_{\sigma_n}\Lambda(g_{\sigma_n})k_{\sigma_n}^{-1}) \\
&\quad = k_{\sigma_0}\Lambda(g_{\sigma_0}h_{e_1}g_{\sigma_1}\cdots h_{e_n}g_{\sigma_n})k_{\sigma_n}^{-1} \\
&\quad = \Lambda \circ \pi_1(\phi_1, \sigma_0)(x)
\end{aligned}$$

since $k_{\sigma_0} = k_{\sigma_n} = 1$. Thus (1) commutes.

To prove that (2) commutes, let

$$\tilde{L} = \tilde{L}_{T'} \circ L^{\lambda}_{T,T'} \circ \tilde{L}_T^{-1}$$

We will show that $\tilde{L} = L$. By the equivariance of the morphisms of scwols used to define $\tilde{L}$, and the commutativity of (1), we have that $\tilde{L}$ is Λ-equivariant. Thus it is enough to check (for example) that $\tilde{L}(h_{i(\alpha)}\overline{\alpha}) = L(h_{i(\alpha)}\overline{\alpha})$ for all $\alpha \in \mathcal{Y}$. By Proposition 27,

$$\begin{aligned}
\tilde{L}(h_{i(\alpha)}\overline{\alpha}) &= \tilde{L}_{T'} \circ L^{\lambda}_{T,T'}([1], \alpha) \\
&= \tilde{L}_{T'}([u_{i(\alpha)}], l(\alpha)) \\
&= \Lambda_{T'}(u_{i(\alpha)})h_{i(l(\alpha))}\overline{l(\alpha)}.
\end{aligned}$$

Let $\pi_{i(\alpha)} = e_1 e_2 \cdots e_n$ be the element of $FG(\mathcal{Y})$ which corresponds to the non-backtracking path in T from σ_0 to $i(\alpha)$, and similarly for $\pi'_{i(l(\alpha))} = e'_1 e'_2 \cdots e'_{n'}$ in $FG'(\mathcal{Y}')$. Then

$$\begin{aligned}
\Lambda_{T'}(u_{i(\alpha)}) &= \Lambda_{T'} \circ \kappa'^{-1}_{T'} \left\{ F\lambda(\pi_{i(\alpha)})(\pi'_{i(l(\alpha))})^{-1} \right\} \\
&= \pi_1(\phi'_1, \sigma'_0) \left\{ F\lambda(\pi_{i(\alpha)})(\pi'_{i(l(\alpha))})^{-1} \right\} \\
&= \pi_1(\phi'_1, \sigma'_0) \left\{ F\lambda(e_1)F\lambda(e_2)\cdots F\lambda(e_n)e'^{-1}_{n'}\cdots e'^{-1}_2 e'^{-1}_1 \right\} \\
&= k_{\sigma_0}\Lambda(h_{e_1})k_{\sigma_1}^{-1}k_{\sigma_1}\Lambda(h_{e_2})k_{\sigma_2}^{-1}\cdots k_{\sigma_{n-1}}\Lambda(h_{e_n})k_{\sigma_n}^{-1}h_{e'_{n'}}^{-1}\cdots h_{e'_2}^{-1}h_{e'_1}^{-1} \\
&= k_{\sigma_0}\Lambda(h_{e_1}h_{e_2}\cdots h_{e_n})k_{\sigma_n}^{-1}(h_{e'_1}h_{e'_2}\cdots h_{e'_{n'}})^{-1} \\
&= \Lambda(h_{i(\alpha)})k_{\sigma_n}^{-1}h_{i(l(\alpha))}^{-1}
\end{aligned}$$

since $k_{\sigma_0} = 1$. Substituting, we obtain finally

$$\begin{aligned}
\tilde{L}(h_{i(\alpha)}\overline{\alpha}) &= \Lambda(h_{i(\alpha)})k_{\sigma_n}^{-1}\overline{l(\alpha)} \\
&= \Lambda(h_{i(\alpha)})k_{i(\alpha)}^{-1}\overline{l(\alpha)} \\
&= \Lambda(h_{i(\alpha)})L(\overline{\alpha}) \\
&= L(h_{i(\alpha)}\overline{\alpha})
\end{aligned}$$

as desired. This completes the proof of the Main Lemma.

The following result makes precise the relationship between a developable complex of groups $G(\mathcal{Y})$ and the complex of groups induced by the action of $\pi_1(G(\mathcal{Y}), T)$ on $D(\mathcal{Y}, T)$, for some maximal tree T in $\mathcal{Y}$. It will be used to prove the Corollary to the Main Lemma below.

Lemma 41 *Let $G(\mathcal{Y})$ be a developable complex of groups over a connected scwol $\mathcal{Y}$. Choose a vertex $\sigma_0 \in V(\mathcal{Y})$ and a maximal tree T in $\mathcal{Y}$. Let $\mathcal{Z}$ be the quotient scwol*

$$\mathcal{Z} = \pi_1(G(\mathcal{Y}), T)\backslash D(\mathcal{Y}, T)$$

and let f be the canonical isomorphism of scwols

$$\begin{aligned}
f : \mathcal{Y} &\to \mathcal{Z} \\
\alpha &\mapsto \pi_1(G(\mathcal{Y}), T) \cdot ([1], \alpha)
\end{aligned}$$

Let $C_\bullet$ be the following data for the action of $\pi_1(G(\mathcal{Y}), T)$ on $D(\mathcal{Y}, T)$:

$$\overline{f(\alpha)} = ([1], \alpha) \quad \textit{and} \quad h_{f(a)} = a^+$$

and let $G(\mathcal{Z})_{C_\bullet}$ be the complex of groups associated to this data. Then there is an isomorphism of complexes of groups

$$\theta : G(\mathcal{Y}) \to G(\mathcal{Z})$$

over f such that

$$\Lambda^\theta_{T,f(T)} = \Lambda^{-1}_{f(T)} \quad \textit{and} \quad L^\theta_{T,f(T)} = \tilde{L}^{-1}_{f(T)}$$

where $f(T)$ is the image of T in $\mathcal{Z}$.

PROOF. We define θ by $\theta_\sigma(g) = g$ for each $g \in G_\sigma$, and $\theta(a) = 1$ for each $a \in E(\mathcal{Y})$ (here we are identifying G_σ with its image in $\pi_1(G(\mathcal{Y}), T)$).

We then have

$$\Lambda^\theta_{T,f(T)} \circ \Lambda_{f(T)} = \kappa^{-1}_{f(T)} \circ \pi_1(\theta, \sigma_0) \circ \kappa_T \circ \pi_1(\phi_1, f(\sigma_0)) \circ \kappa_{f(T)}.$$

We claim that

$$\pi_1(\theta, \sigma_0) \circ \kappa_T \circ \pi_1(\phi_1, f(\sigma_0)) = 1. \tag{2}$$

Let $g \in \pi_1(G(\mathcal{Z}), f(\sigma_0))$. Then $g = g_0 f(e_1) g_1 \cdots f(e_n) g_n$ for some $G(\mathcal{Z})$-loop $(g_0, f(e_1), g_1, \ldots, f(e_n), g_n)$ based at $f(\sigma_0) = f(\sigma_n)$, and so

$$\begin{aligned}
\kappa_T \circ \pi_1(\phi_1, f(\sigma_0))(g) &= \kappa_T(g_0 h_{f(e_1)} g_1 \cdots h_{f(e_n)} g_n) \\
&= \pi_{\sigma_0} g_0 \pi_{\sigma_0}^{-1} \kappa_T(h_{f(e_1)}) \pi_{\sigma_1} g_1 \pi_{\sigma_1}^{-1} \cdots \kappa_T(h_{f(e_n)}) \pi_{\sigma_n} g_n \pi_{\sigma_n}^{-1}
\end{aligned}$$

where π_σ is the unique non-backtracking path in T from σ_0 to σ. Now, applying $h_{f(a)} = a^+$ and $\kappa_T(a^+) = \pi_{t(a)} a^+ \pi_{i(a)}^{-1}$, as well as $\pi_{\sigma_0} = \pi_{\sigma_n} = 1$, we have

$$\begin{aligned}
\pi_1(\theta, \sigma_0) \circ \kappa_T \circ \pi_1(\phi_1, f(\sigma_0))(g) &= \pi_1(\theta, \sigma_0)(g_0 e_1 g_1 \cdots e_n g_n) \\
&= g_0 f(e_1) g_1 \cdots f(e_n) g_n \\
&= g
\end{aligned}$$

and so Equation (2) holds. Thus $\Lambda^\theta_{T,f(T)} \circ \Lambda_{f(T)} = 1$. By conjugating Equation (2), we obtain

$$\Lambda_{f(T)} \circ \Lambda^\theta_{T,f(T)} = 1$$

and conclude that $\Lambda^\theta_{T,f(T)} = \Lambda^{-1}_{f(T)}$.

To show that $L^\theta_{T,f(T)} = \tilde{L}^{-1}_{f(T)}$, let

$$u_\sigma = \kappa^{-1}_{f(T)} \left\{ F\theta(\pi_\sigma) \left(\pi'_{f(\sigma)}\right)^{-1} \right\}$$

be the elements of $\pi_1(G(\mathcal{Z}), f(T))$ with respect to which $L^\theta_{T,f(T)}$ is defined. Here π_σ denotes the non-backtracking path in T from σ_0 to σ, and similarly for $\pi'_{f(\sigma)}$ and $f(T)$. By definition of θ,

$$F\theta(\pi_\sigma) = \pi'_{f(\sigma)}$$

hence $u_\sigma = 1$ for all $\sigma \in V(\mathcal{Y})$. Also, for each $\alpha \in \mathcal{Y}$, the element $h_{i(f(\alpha))} \in \pi_1(G(\mathcal{Y}), T)$ with respect to which $\tilde{L}_{f(T)}$ is defined is a product of oriented edges $a^\pm$ with $a \in T$. Hence $h_{i(f(\alpha))} = 1$.

Applying these facts, we have, for $g \in \pi_1(G(\mathcal{Y}), T)$ and $\alpha \in \mathcal{Y}$,

$$\begin{aligned}
\tilde{L}_{f(T)} \circ L^\theta_{T,f(T)}([g], \alpha) &= \tilde{L}_{f(T)}([\Lambda^\theta_{T,f(T)}(g)], f(\alpha)) \\
&= \Lambda_{f(T)} \circ \Lambda^\theta_{T,f(T)}(g) h_{i(f(\alpha))} \overline{f(\alpha)} \\
&= g([1], \alpha) \\
&= ([g], \alpha)
\end{aligned}$$

and

$$\begin{aligned} L^{\theta}_{T,f(T)} \circ \tilde{L}_{f(T)}([g], f(\alpha)) &= L^{\theta}_{T,f(T)}(\Lambda_{f(T)}(g)\overline{f(\alpha)}) \\ &= L^{\theta}_{T,f(T)}([\Lambda_{f(T)}(g)], \alpha) \\ &= ([\Lambda^{\theta}_{T,f(T)} \circ \Lambda_{f(T)}(g)], f(\alpha)) \\ &= ([g], f(\alpha)). \end{aligned}$$

Thus $L^{\theta}_{T,f(T)} = \tilde{L}^{-1}_{f(T)}$.

The following result corresponds to Corollary 4.5, [1].

Corollary 42 (Corollary to the Main Lemma) *Let $G(\mathcal{Y})$ and $G'(\mathcal{Y}')$ be developable complexes of groups over connected scwols $\mathcal{Y}$ and $\mathcal{Y}'$, and choose maximal trees T and T' in $\mathcal{Y}$ and $\mathcal{Y}'$ respectively. Suppose $L : D(\mathcal{Y}, T) \to D(\mathcal{Y}', T')$ is a morphism of scwols which is equivariant with respect to some homomorphism of groups $\Lambda : \pi_1(G(\mathcal{Y}), T) \to \pi_1(G'(\mathcal{Y}'), T')$. If there is a $\sigma_0 \in V(\mathcal{Y})$ such that*

$$L([1], \sigma_0) = ([1], \sigma_0')$$

for some $\sigma_0' \in V(\mathcal{Y}')$, then there exists a morphism $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ of complexes of groups such that $L = L^{\lambda}_{T,T'}$ and $\Lambda = \Lambda^{\lambda}_{T,T'}$.

PROOF. Let the quotient scwol $\mathcal{Z}$, the isomorphism $f : \mathcal{Y} \to \mathcal{Z}$, the data $C_\bullet$, the complex of groups $G(\mathcal{Y})_{C_\bullet}$ and the isomorphism $\theta : G(\mathcal{Y}) \to G(\mathcal{Z})$ be as in the statement of Lemma 41 above, and similarly for $\mathcal{Z}'$, f', $C'_\bullet$, $G'(\mathcal{Y}')_{C'_\bullet}$ and θ'. Let $l : \mathcal{Z} \to \mathcal{Z}'$ be the map of quotient scwols induced by L and Λ. By definition of l, $C_\bullet$ and $C'_\bullet$, and by the assumption on L, we have

$$L(\overline{\sigma_0}) = \overline{l(\sigma_0)}$$

so we may choose $N_\bullet$ with $k_{\sigma_0} = 1$. Let

$$\mu = \mu_{C_\bullet, C'_\bullet, N_\bullet} : G(\mathcal{Z})_{C_\bullet} \to G'(\mathcal{Z}')_{C'_\bullet}$$

be the induced morphism of complexes of groups.

Let

$$\lambda = \theta'^{-1} \circ \mu \circ \theta : G(\mathcal{Y}) \to G'(\mathcal{Y}').$$

We claim that $\Lambda = \Lambda^{\lambda}_{T,T'}$ and $L = L^{\lambda}_{T,T'}$. By Corollary 32, it is enough to show that

$$\Lambda = (\Lambda^{\theta'}_{T',f'(T')})^{-1} \circ \Lambda^{\mu}_{f(T),f'(T')} \circ \Lambda^{\theta}_{T,f(T)}$$

and

$$L = (L^{\theta'}_{T',f'(T')})^{-1} \circ L^{\mu}_{f(T),f'(T')} \circ L^{\theta}_{T,f(T)}.$$

The result follows from the Main Lemma applied to μ, and Lemma 41 above.

This section considers the relationship between the existence of a covering and developability.

Lemma 43 *Let $G(\mathcal{Y})$ and $G'(\mathcal{Y}')$ be complexes of groups over nonempty, connected scwols $\mathcal{Y}$ and $\mathcal{Y}'$. Assume there is a covering $\phi : G(\mathcal{Y}) \to G'(\mathcal{Y}')$. If $G'(\mathcal{Y}')$ is developable, then $G(\mathcal{Y})$ is developable.*

PROOF. Let $\iota' : G'(\mathcal{Y}') \to FG'(\mathcal{Y}')$ be the natural morphism defined after Definition 18 in Section 2.4.2. By Proposition 19, since $G'(\mathcal{Y}')$ is developable, ι' is injective on the local groups. Thus, as ϕ is a covering, the composite morphism $\iota' \circ \phi : G(\mathcal{Y}) \to FG'(\mathcal{Y}')$ is injective on the local groups. Hence, by Proposition 15, the complex of groups $G(\mathcal{Y})$ is developable.

We do not know if the converse to Lemma 43 holds in general. However, in the presence of nonpositive curvature, we have the following partial converse to Lemma 43. Recall that an M_κ-polyhedral complex is a polyhedral complex with n-dimensional cells isometric to polyhedra in the simply connected Riemannian n-manifold of constant sectional curvature κ.

Lemma 44 *Let $\phi : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a covering of complexes of groups, over a morphism of scwols $l : \mathcal{Y} \to \mathcal{Y}'$. Suppose that for some $\kappa \leq 0$, $\mathcal{Y}$ and $\mathcal{Y}'$ are the scwols associated to connected M_κ-polyhedral complexes with finitely many isometry classes of cells K and K' respectively, and that $|l| : |\mathcal{Y}| \to |\mathcal{Y}'|$ is a local isometry on each simplex. If $G(\mathcal{Y})$ has nonpositive curvature (thus is developable), then $G'(\mathcal{Y}')$ also has nonpositive curvature, thus $G'(\mathcal{Y}')$ is developable.*

PROOF. By Lemma 30, to show that $G'(\mathcal{Y}')$ is nonpositively curved, it suffices to show that for each vertex τ' of K', the geometric link of $\tilde{\tau}'$ in the local development $\mathrm{st}(\tilde{\tau}')$, with the induced spherical structure, is CAT(1). We first show, using the following lemma, that if τ' is a vertex of K', then $\tau' = f(\tau)$ for some vertex τ of K.

Lemma 45 *The nondegenerate morphism of scwols $l : \mathcal{Y} \to \mathcal{Y}'$ associated to the covering $\phi : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ surjects onto the set of vertices of $\mathcal{Y}'$.*

PROOF. Let $\sigma \in V(\mathcal{Y})$ and $l(\sigma) = \sigma' \in V(\mathcal{Y}')$. From the definitions of nondegenerate morphism of scwols and covering of complexes of groups, it follows that every vertex of $\mathcal{Y}'$ which is incident to an edge meeting σ' lies in the image of l. Since $\mathcal{Y}'$ is connected, we conclude that l surjects onto $V(\mathcal{Y}')$.

Let τ' be a vertex of K'. By Lemma 45, $\tau' = l(\tau)$ for some $\tau \in V(\mathcal{Y})$. Suppose τ is not a vertex of K. Then there is an $a \in E(\mathcal{Y})$ such that $i(a) = \tau$. It follows that $i(l(a)) = l(i(a)) = \tau'$, so $l(a) \in E(\mathcal{Y}')$ has initial vertex τ'. This contradicts τ' a vertex of K'. Hence τ is a vertex of K.

Since $G(\mathcal{Y})$ is nonpositively curved, the geometric link of $\tilde{\tau}$ in the local development $\mathrm{st}(\tilde{\tau})$, with the induced spherical structure, is CAT(1). By Lemma 28, there is a ϕ_τ-equivariant bijection $\mathrm{st}(\tilde{\tau}) \to \mathrm{st}(\tilde{\tau}')$. We claim this bijection is an isometry in the induced metric, which completes the proof.

By definition of the induced metric, the action of G_τ on $\mathrm{st}(\tilde{\tau})$ induces a simplicial map $\mathrm{st}(\tilde{\tau}) \to \mathrm{st}(\tau)$ which is a local isometry on each simplex. Similarly, the action of $G_{\tau'}$ on $\mathrm{st}(\tilde{\tau}')$ induces $\mathrm{st}(\tilde{\tau}) \to \mathrm{st}(\tau)$ which is a local isometry on each simplex. By assumption, the restriction of $|l|$ to $\mathrm{st}(\tau)$ is a local isometry on each simplex. Hence, the bijection $\mathrm{st}(\tilde{\tau}) \to \mathrm{st}(\tilde{\tau}')$ is a local isometry on each simplex, and thus an isometry.

4 The Conjugacy Theorem for Complexes of Groups

In this section, we prove the analogue for complexes of groups of the Conjugacy Theorem for graphs of groups (Theorem 5.2 of [1]). Let us prove the following lemma which characterizes coverings.

Lemma 46 (Corollary 4.6, [1]) *With the notation in Definition 17, the induced morphism λ is a covering if and only if Λ is a monomorphism and L is an isomorphism.*

PROOF. By the Main Lemma in Section 3, since the vertical maps are isomorphisms, Λ is a monomorphism if and only if $\Lambda^\lambda_{T,T'}$ is a monomorphism, and L an isomorphism if and only if $L^\lambda_{T,T'}$ is an isomorphism.

Suppose λ is a covering. Then by Proposition 33, $\Lambda^\lambda_{T,T'}$ is a monomorphism and $L^\lambda_{T,T'}$ is an isomorphism, and the conclusion follows.

Conversely, suppose Λ is a monomorphism and L is an isomorphism. Assume by contradiction that λ is not a covering. Then either

(1) there is a homomorphism $\lambda_\sigma : G_\sigma \to G'_{l(\sigma)}$ which is not injective, or

(2) there exists $a' \in E(\mathcal{Y}')$ and $\sigma \in V(\mathcal{Y})$ with $t(a') = \sigma' = l(\sigma)$, such that the map

$$\coprod_{\substack{a \in l^{-1}(a') \\ t(a)=\sigma}} G_\sigma/\psi_a(G_{i(a)}) \to G'_{\sigma'}/\psi_{a'}(G'_{i(a')})$$

induced by

$$g \mapsto \lambda_\sigma(g)\lambda(a)$$

is not bijective.

Condition (1) implies that the map $\Lambda^\lambda_{T,T'}$ is not a monomorphism at G_σ, thus $\Lambda^\lambda_{T,T'}$ is not a monomorphism. Condition (2) implies that $L^\lambda_{T,T'}$ is not a local bijection at $\mathrm{St}(\tilde{\sigma})$ (see Remark 5.3, [4]), thus the map $L^\lambda_{T,T'}$ is not an isomorphism. By contradiction, we conclude that λ is a covering.

Let $\mathcal{X}$ be the scwol associated to a polyhedral complex K. Let $G = \mathrm{Aut}(K)$, and let H be a subgroup of G acting without inversions. Then H acts on $\mathcal{X}$ in the sense of Definition 10. Define

$$G_H = \{g \in G \mid g\sigma \in H\sigma \text{ for all } \sigma \in V(\mathcal{X})\}.$$

Then G_H is a subgroup of $\mathrm{Aut}(K)$, H is a subgroup of G_H and

$$H\backslash\mathcal{X} = G_H\backslash\mathcal{X}.$$

The following theorem is the same as the Conjugacy Theorem stated in the Introduction.

Theorem 47 *If $\Gamma \leq G_H$ acts freely on $\mathcal{X}$ then there is an element $g \in G_H$ such that $g\Gamma g^{-1} \leq H$.*

PROOF. Let $\mathcal{A} = H\backslash\mathcal{X} = G_H\backslash\mathcal{X}$ and $\mathcal{B} = \Gamma\backslash\mathcal{X}$ and let $f : \mathcal{B} \to \mathcal{A}$ be the natural projection (coming from $\Gamma \leq G_H$). We form quotient complexes of groups $G(\mathcal{A}) = (G_\sigma, \psi_a, g_{a,b})$ induced by the action of H and $G'(\mathcal{A}) = (G'_\sigma, \psi'_a, g'_{a,b})$ induced by the action of G_H, using the same maximal tree $T_\mathcal{A}$ in the one-skeleton of $\mathcal{A}$ and the same family of elements $h_a \in H \leq G_H$, for $a \in E(\mathcal{A})$. Then for each $\sigma \in V(\mathcal{A})$ we have $G_\sigma \leq G'_\sigma$, and for each edge $a \in E(\mathcal{A})$ we have $\psi'_a|_{G_{i(a)}} = \psi_a$. There is then by Corollary 46 a covering morphism

$$\lambda : G(\mathcal{A}) \to G'(\mathcal{A})$$

induced by the identity map $L : \mathcal{A} \to \mathcal{A}$ and the inclusion $\Lambda : H \to G_H$. By definition of induced morphism, each λ_σ is inclusion, and each $\lambda(a)$ is trivial. Hence for all $a \in E(\mathcal{A})$, the inclusion induced map

$$G_{t(a)}/\psi_a(G_{i(a)}) \to G'_{t(a)}/\psi'_a(G'_{i(a)})$$

given by $[g] \mapsto [g]$ is a bijection.

Now form the quotient complex of (trivial) groups (since Γ acts freely), $G''(\mathcal{B}) = (G''_\sigma, \psi''_a, g''_{a,b})$ and let $\phi = (\phi_\sigma, \phi(b)) : G''(\mathcal{B}) \to G'(\mathcal{A})$ be a covering morphism induced by the inclusion $\Gamma \leq G_H$ over the natural projection $f : \mathcal{B} \to \mathcal{A}$. For each $a \in E(\mathcal{A})$ with $t(a) = f(\tau)$, there is a bijection

$$\{b \in f^{-1}(a) \mid t(b) = \tau\} \to G'_{t(a)}/\psi'_a(G'_{i(a)})$$

given by $b \mapsto [\phi(b)]$ where since $G''(\mathcal{B})$ is a complex of trivial groups, we replace the one-element sets $G''_\tau/\psi''_b(G''_{i(b)})$ by $\{b\}$. Thus since λ is a covering, for each $b \in E(\mathcal{B}')$, with $f(b) = a$, we can find elements $g_b \in G_{t(a)}$ such that

$$[\phi(b)] = [g_b]$$

in $G'_{t(a)}/\psi'_a(G'_{i(a)})$.

We now define a morphism $\phi' : G''(\mathcal{B}) \to G(\mathcal{A})$ over f, by each ϕ'_σ being the inclusion of the trivial group, and $\phi'(b) \in G_{t(f(b))}$ being $\phi'(b) = g_b$. We then have a bijection

$$\{b \in f^{-1}(a) \mid t(b) = \tau\} \to G_{t(a)}/\psi_a(G_{i(a)})$$

given by $b \mapsto [g_b]$ hence ϕ' is a covering.

Choose a maximal tree $T_\mathcal{B}$ in $\mathcal{B}$ and recall that we chose a maximal tree $T_\mathcal{A}$ in $\mathcal{A}$. By Proposition 33 the covering $\phi' : G''(\mathcal{B}) \to G(\mathcal{A})$ induces a monomorphism of groups

$$\Lambda^{\phi'}_{T_\mathcal{B},T_\mathcal{A}} : \pi_1(G''(\mathcal{B}), T_\mathcal{B}) \to \pi_1(G(\mathcal{A}), T_\mathcal{A})$$

and a $\Lambda_{T_\mathcal{B},T_\mathcal{A}}$–equivariant isomorphism of scwols

$$L^{\phi'}_{T_\mathcal{B},T_\mathcal{A}} : D(G''(\mathcal{B}), T_\mathcal{B}) \to D(G(\mathcal{A}), T_\mathcal{A})$$

such that the following diagram commutes:

$$\begin{tikzcd}
D(G''(\mathcal{B}), T_\mathcal{B}) \arrow[r, "L^{\phi'}_{T_\mathcal{B},T_\mathcal{A}}"] \arrow[d] & D(G(\mathcal{A}), T_\mathcal{A}) \arrow[d] \\
\mathcal{B} \arrow[r, "f"] & \mathcal{A}
\end{tikzcd}$$

where the vertical arrows are the natural projections. Let $\widetilde{L}_{T_\mathcal{B}} : D(G''(\mathcal{B}), T_\mathcal{B}) \to \mathcal{X}$ and $\widetilde{L}_{T_\mathcal{A}} : D(G(\mathcal{A}), T_\mathcal{A}) \to \mathcal{X}$ be the canonical isomorphisms, equivariant with respect to the isomorphisms of groups $\Lambda_{T_\mathcal{B}} : \pi_1(G''(\mathcal{B}), T_\mathcal{B}) \to \Gamma$ and $\Lambda_{T_\mathcal{A}} : \pi_1(G(\mathcal{A}), T_\mathcal{A}) \to H$, respectively. Let $g \in \mathrm{Aut}(\mathcal{X})$ be the following composition of isomorphisms

$$g = \widetilde{L}_{T_{\mathcal{A}}} \circ L^{\phi'}_{T_{\mathcal{B}},T_{\mathcal{A}}} \circ \widetilde{L}^{-1}_{T_{\mathcal{B}}} : \mathcal{X} \to \mathcal{X}.$$

Then g is equivariant with respect to the monomorphism $\theta : \Gamma \to H$ given by

$$\theta = \Lambda_{T_{\mathcal{A}}} \circ \Lambda^{\phi'}_{T_{\mathcal{B}},T_{\mathcal{A}}} \circ \Lambda^{-1}_{T_{\mathcal{B}}} : \Gamma \to H.$$

Thus for all $\gamma \in \Gamma \leq \mathrm{Aut}(\mathcal{X})$, we have

$$g \circ \gamma = \theta(\gamma) \circ g$$

and so $g \circ \gamma \circ g^{-1} = \theta(\gamma) \leq H$. That is, $g\Gamma g^{-1} \leq H$.

It remains to show that $g \in G_H$. Let $p : \mathcal{X} \to \mathcal{A} = H\backslash\mathcal{X} = G_H\backslash\mathcal{X}$ and $p_\Gamma : \mathcal{X} \to \mathcal{B} = \Gamma\backslash\mathcal{X}$ be the natural projections. Then $p = f \circ p_\Gamma$. We wish to show that $p \circ g = p$. Now p_Γ is the composition of $\widetilde{L}^{-1}_{T_{\mathcal{B}}} : X \to D(G''(\mathcal{B}), T_{\mathcal{B}})$ with the natural projection $D(G''(\mathcal{B}), T_{\mathcal{B}}) \to \mathcal{B}$, and similarly p is the composition of $\widetilde{L}^{-1}_{T_{\mathcal{A}}} : \mathcal{X} \to D(G(\mathcal{A}), T_{\mathcal{A}})$ with the natural projection $D(G(\mathcal{A}), T_{\mathcal{A}}) \to \mathcal{A}$. Hence the definition of g and the commutativity of the diagram above mean that $p \circ g = f \circ p_\Gamma = p$ as required.

5 Coverings and overgroups

In this section we prove Theorem 4, stated in the Introduction. We first define isomorphism of coverings. In Section 5.1 we define a map from overgroups to coverings, and in Section 5.2 a map from coverings to overgroups. Then in Section 5.3 we conclude the proof of Theorem 4 by showing that these maps are mutual inverses.

Definition 48 (isomorphism of coverings) *Let $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ and $\lambda' : G(\mathcal{Y}) \to G''(\mathcal{Y}'')$ be coverings of developable complexes of groups over connected scwols. Fix $\sigma_0 \in V(\mathcal{Y})$. We say that λ and λ' are isomorphic coverings if for any choice of maximal trees T, T' and T'' in $\mathcal{Y}$, $\mathcal{Y}'$ and $\mathcal{Y}''$ respectively, there exists an isomorphism $\lambda'' : G'(\mathcal{Y}') \to G''(\mathcal{Y}'')$ of complexes of groups such that the following diagram of morphisms of universal covers (defined in Proposition 31) commutes*

$$\begin{tikzcd} D(\mathcal{Y},T) \arrow[r, "L^{\lambda}_{T,T'}"] \arrow[dr, "L^{\lambda'}_{T,T''}"'] & D(\mathcal{Y}',T') \arrow[d, "L^{\lambda''}_{T',T''}"] \\ & D(\mathcal{Y}'',T''). \end{tikzcd}$$

Note that by Corollary 32, this diagram commutes for one triple (T, T', T'') if and only if it commutes for all triples (T, T', T''). By Proposition 33, since λ and λ' are coverings, $L^{\lambda}_{T,T'}$ and $L^{\lambda'}_{T,T''}$ are isomorphisms. By Proposition 39, since λ'' is an isomorphism, the map $L^{\lambda''}_{T',T''}$ is an isomorphism. Hence, two coverings are isomorphic if and only if they induce a commutative diagram of isomorphisms of universal covers.

For the remainder of Section 5, we fix the following data:

- $\mathcal{X}$, the scwol associated to a simply connected polyhedral complex K,
- Γ, a subgroup of $\mathrm{Aut}(K)$ which acts on $\mathcal{X}$, with quotient $\mathcal{Y} = \Gamma\backslash\mathcal{X}$,
- a vertex $\sigma_0 \in V(\mathcal{Y})$, and
- a set of choices $C_\bullet = (\overline{\sigma}, h_a)$ giving rise to a complex of groups $G(\mathcal{Y})_{C_\bullet} = (G_\sigma, \psi_a, g_{a,b})$ induced by the action of Γ on $\mathcal{X}$.

Let $\mathrm{Over}(\Gamma)$ be the set of overgroups of Γ which act without inversions, that is, the set of subgroups of $\mathrm{Aut}(K)$ containing Γ which act without inversions. Let $\mathrm{Cov}(G(\mathcal{Y}))$ be the set of isomorphism classes of coverings of faithful, developable complexes of groups by $G(\mathcal{Y})$.

5.1 The map from overgroups to coverings

In this section we construct a map

$$\underline{a} : \mathrm{Over}(\Gamma) \to \mathrm{Cov}(G(\mathcal{Y})).$$

We first show in Lemma 49 that an overgroup induces a covering of complexes of groups. Then in Lemma 50 we show that, without loss of generality, we may apply the Main Lemma to this covering. In Lemma 51, we define $\underline{a}$ and show that $\underline{a}$ is well-defined on isomorphism classes of coverings.

Lemma 49 *Let Γ' be an overgroup of Γ acting without inversions. Let $G'(\mathcal{Y}')_{C'_\bullet}$ be a complex of groups over $\mathcal{Y}' = \Gamma'\backslash\mathcal{X}$ induced by the action of Γ' on $\mathcal{X}$, for some choices $C'_\bullet$. Let $L = Id : \mathcal{X} \to \mathcal{X}$ and let $\Lambda : \Gamma \hookrightarrow \Gamma'$ be inclusion, inducing $l : \mathcal{Y} \to \mathcal{Y}'$. For some choices $N_\bullet$, let*

$$\lambda = \lambda_{C_\bullet, C'_\bullet, N_\bullet} : G(\mathcal{Y})_{C_\bullet} \to G'(\mathcal{Y}')_{C'_\bullet}$$

be the morphism of complexes of groups over l induced by L and Λ (see Definition 17). Then λ is a covering.

PROOF. By definition, $\lambda_\sigma = \mathrm{Ad}(k_\sigma)$, where $k_\sigma : \overline{\sigma} \mapsto \overline{l(\sigma)}$. The local maps λ_σ are thus injective.

We write $[g]_a$ for the coset of $g \in G_{t(a)}$ in $G_{t(a)}/\psi_a(G_{i(a)})$, and similarly for $[g']_{a'}$ when $g' \in G'_{t(a')}$. It now suffices to show that for every $a' \in E(\mathcal{Y}')$ with $t(a') = \sigma' = l(\sigma) \in V(\mathcal{Y})$, the map on cosets

$$\coprod_{\substack{a \in l^{-1}(a') \\ t(a)=\sigma}} \Gamma_{\overline{\sigma}}/h_a(\Gamma_{\overline{i(a)}})h_a^{-1} \longrightarrow \Gamma'_{\overline{l(\sigma)}}/h'_{a'}(\Gamma'_{\overline{i(a')}})h'^{-1}_{a'}$$

$$[g]_a \longmapsto [\lambda_\sigma(g)\lambda(a)]_{a'}$$

is bijective. Suppose $[\lambda_\sigma(g)\lambda(a)]_{a'} = [\lambda_\sigma(h)\lambda(b)]_{a'}$. Then by definition of λ,

$$h'_{a'}k_{i(b)}h_b^{-1}k_{t(b)}^{-1}k_\sigma h^{-1}gk_\sigma^{-1}k_{t(a)}h_ak_{i(a)}^{-1}(h'_{a'})^{-1} \in h'_{a'}G'_{i(a')}(h'_{a'})^{-1}$$

hence

$$k_{i(b)}h_b^{-1}h^{-1}gh_ak_{i(a)}^{-1} \in \Gamma'_{\overline{i(a')}}.$$

Since $k_{i(a)}$ and $k_{i(b)}$ send $\overline{i(a)}$ and $\overline{i(b)}$ respectively to $\overline{i(a')}$, the element $h_b^{-1}h^{-1}gh_a$ in Γ sends $\overline{i(a)}$ to $\overline{i(b)}$. Since $l(a) = l(b)$, this implies that $a = b$. Hence $h^{-1}g$ maps $\overline{i(a)}$ to itself, thus $[h]_a = [g]_a$. Therefore the map on cosets is injective.

Let us show that the map on cosets is surjective. Let $[h']_{a'}$ be an element of the target set. Let $b' = k_\sigma^{-1}h'h'_{a'}(\overline{a'})$. Since $h' \in \Gamma'_{\overline{\sigma'}}$, we have $t(b') = \overline{\sigma}$. Let $c = p(b')$, where p is the natural projection $\mathcal{X} \to \mathcal{Y} = \Gamma\backslash\mathcal{X}$. Let $g \in \Gamma_{\overline{\sigma}}$ be such that $g(h_c\overline{c}) = b'$. We claim that $[g]_c$ maps to $[h']_{a'}$, that is,

$$h'^{-1}k_\sigma gh_ck_{i(c)}^{-1}h'^{-1}_{a'} \in h'_{a'}\Gamma'_{\overline{i(a')}}h'^{-1}_{a'}.$$

Since $k_{i(c)}^{-1}$ sends $\overline{i(a')}$ to $\overline{i(c)}$, and the element $k_\sigma gh_c$ sends $\overline{i(c)}$ to $i(k_\sigma b') = i(h'h'_{a'}(\overline{a'}))$, it follows that $h'^{-1}_{a'}h'^{-1}k_\sigma gh_ck_{i(c)}^{-1}$ fixes $\overline{i(a')}$, which proves the claim.

We now show that every covering λ induced by an overgroup, as in Lemma 49, is isomorphic to a covering λ' to which the Main Lemma may be applied. More precisely:

Lemma 50 *With the notation of Lemma 49, fix a vertex $\sigma_0 \in V(\mathcal{Y})$. Then there is a choice $C''_\bullet$ of data for Γ' acting on $\mathcal{X}$ such that $\overline{\sigma_0} = \overline{l(\sigma_0)}$, and a choice $N'_\bullet = \{k'_\sigma\}$ such that $k'_{\sigma_0} = 1$, so that λ is isomorphic to the covering*

$$\lambda' = \lambda'_{C_\bullet, C''_\bullet, N'_\bullet} : G(\mathcal{Y})_{C_\bullet} \to G''(\mathcal{Y}'')_{C''_\bullet}$$

where $G''(\mathcal{Y}'')_{C''_\bullet}$ is the complex of groups induced by $C''_\bullet$.

PROOF. By definition of l, there is a choice $C''_\bullet$ so that $\overline{\sigma_0}$, determined by $C_\bullet$, equals $\overline{l(\sigma_0)}$ determined by $C''_\bullet$. We now define a collection $N'_\bullet = \{k'_\sigma\}$ such that $k'_\sigma\overline{\sigma} = \overline{l(\sigma)}$ for all $\sigma \in V(\mathcal{Y})$.

Choose a section $s : V(\mathcal{Y}') \to V(\mathcal{Y})$ for l. That is, for each $\sigma' \in V(\mathcal{Y}')$, choose $s(\sigma') \in V(\mathcal{Y})$ such that $l(s(\sigma')) = \sigma'$. In particular, if $\sigma'_0 = l(\sigma_0)$, let $s(\sigma'_0) = \sigma_0$.

For each $s(\sigma') \in V(\mathcal{Y})$, choose an element $k'_{s(\sigma')} \in \Gamma'$ such that $k'_{s(\sigma')}\overline{s(\sigma')} = \overline{\sigma'}$, where $\overline{s(\sigma')}$ is determined by $C_\bullet$ and $\overline{\sigma'}$ by $C''_\bullet$. Since $s(\sigma'_0) = \sigma_0$, and by choice of $C''_\bullet$, we have $k'_{\sigma_0}\overline{\sigma_0} = \overline{l(\sigma_0)} = \overline{\sigma_0}$, so we may choose $k'_{\sigma_0} = 1$. For all other $\sigma \in V(\mathcal{Y})$, let

$$k'_\sigma = k'_{s(l(\sigma))}k^{-1}_{s(l(\sigma))}k_\sigma \tag{3}$$

where $N_\bullet = \{k_\sigma\}$. Note that

$$k'_\sigma\overline{\sigma} = k'_{s(l(\sigma))}k^{-1}_{s(l(\sigma))}k_\sigma\overline{\sigma} = k'_{s(l(\sigma))}k^{-1}_{s(l(\sigma))}\overline{l(\sigma)} = k'_{s(l(\sigma))}\overline{s(l(\sigma))} = \overline{l(s)}.$$

This defines a collection $N'_\bullet = \{k'_\sigma\}$ with $k'_{\sigma_0} = 1$. Let $\lambda' : G(\mathcal{Y})_{C_\bullet} \to G''(\mathcal{Y}'')_{C'_\bullet}$ be the covering induced by $N'_\bullet$.

We now construct an isomorphism of complexes of groups $\mu : G'(\mathcal{Y}') \to G''(\mathcal{Y}'')$ such that the following diagram commutes

$$\begin{tikzcd} G(\mathcal{Y}) \arrow[r, "\lambda"] \arrow[dr, "\lambda'"'] & G'(\mathcal{Y}') \arrow[d, "\mu"] \\ & G''(\mathcal{Y}''). \end{tikzcd} \tag{4}$$

By Corollary 32, it follows that λ is isomorphic to λ'.

Let $f : \mathcal{Y}' \to \mathcal{Y}''$ be the identity map (both $\mathcal{Y}'$ and $\mathcal{Y}''$ are the quotient $\Gamma'\backslash\mathcal{X}$). We choose a collection $N''_\bullet = \{k''_{\sigma'}\}$ of elements of Γ' such that $k''_{\sigma'}\overline{\sigma'} = \overline{f(\sigma')}$ as follows. By Equation (3), if $l(\sigma_1) = l(\sigma_2)$ then $k'_{\sigma_1}k^{-1}_{\sigma_1} = k'_{\sigma_2}k^{-1}_{\sigma_2}$. Given $\sigma' \in V(\mathcal{Y}')$, it is thus well-defined to put

$$k''_{\sigma'} = k'_\sigma k^{-1}_\sigma$$

for any $\sigma \in l^{-1}(\sigma')$. We check

$$k''_{\sigma'}\overline{\sigma'} = k'_\sigma k^{-1}_\sigma\overline{\sigma'} = k'_\sigma\overline{\sigma} = \overline{\sigma'} = \overline{f(\sigma')}$$

as required. Define $\mu = \mu_{C'_\bullet, C''_\bullet, N''_\bullet} : G'(\mathcal{Y}')_{C'_\bullet} \to G''(\mathcal{Y}'')_{C''_\bullet}$. Since $G'(\mathcal{Y}')$ and $G''(\mathcal{Y}'')$ are both associated to the action of Γ' on $\mathcal{X}$, μ is an isomorphism.

By definition of composition of morphisms, for $g \in G_\sigma$ we have

$$\begin{aligned}(\mu \circ \lambda)_\sigma(g) &= \mu_{l(\sigma)} \circ \lambda_\sigma(g) \\ &= \mathrm{Ad}(k''_{l(\sigma)}) \circ \mathrm{Ad}(k_\sigma)(g) \\ &= \mathrm{Ad}(k''_{l(\sigma)}k_\sigma)(g) \\ &= \mathrm{Ad}(k'_\sigma)(g) \\ &= \lambda'_\sigma(g)\end{aligned}$$

and for $a \in E(\mathcal{Y})$

$$\begin{aligned}
(\mu \circ \lambda)(a) &= \mu_{l(t(a))}(\lambda(a))\mu(l(a)) \\
&= \mathrm{Ad}(k''_{l(t(a))})(k_{t(a)} h_a k^{-1}_{i(a)} h^{-1}_{l(a)}) k''_{t(l(a))} h_{l(a)} (k''_{i(l(a))})^{-1} h^{-1}_{f(l(a))} \\
&= k''_{l(t(a))} k_{t(a)} h_a k^{-1}_{i(a)} (k''_{i(l(a))})^{-1} h^{-1}_{f(l(a))} \\
&= k'_{t(a)} h_a (k'_{i(a)})^{-1} h_{f(l(a))} \\
&= \lambda'(a)
\end{aligned}$$

hence the diagram at (4) commutes.

Lemma 51 *Let*

$$\underline{a} : \mathrm{Over}(\Gamma) \to \mathrm{Cov}(G(\mathcal{Y}))$$

be a map taking an overgroup Γ' of Γ to a covering, as described in Lemma 49. Let $C'_\bullet$, $N_\bullet$ and $C''_\bullet$, $N'_\bullet$ be any two choices for the construction of $\underline{a}(\Gamma')$

$$\lambda_{C_\bullet, C'_\bullet, N_\bullet} : G(\mathcal{Y})_{C_\bullet} \to G'(\mathcal{Y}')_{C'_\bullet} \quad \textit{and} \quad \lambda'_{C_\bullet, C''_\bullet, N'_\bullet} : G(\mathcal{Y})_{C_\bullet} \to G''(\mathcal{Y}'')_{C''_\bullet}$$

Then λ and λ' are isomorphic coverings, so $\underline{a}$ is well-defined.

PROOF. Fix a vertex $\sigma_0 \in V(\mathcal{Y})$ and let $\sigma'_0 = l(\sigma_0)$. By Lemma 50, we may without loss of generality assume that the Main Lemma may be applied to λ and λ'. As in the proof of Lemma 50, choose a collection $N''_\bullet = \{k''_{\sigma'}\}$ with $k''_{\sigma'_0} = k'_{\sigma_0} k^{-1}_{\sigma_0} = 1$. Then we may apply the Main Lemma to the isomorphism of complexes of groups

$$\lambda'' = \lambda_{C'_\bullet, C''_\bullet, N''_\bullet} : G'(\mathcal{Y}')_{C'_\bullet} \to G''(\mathcal{Y}'')_{C''_\bullet}.$$

Choose maximal trees T, T' and T'' in $\mathcal{Y}$, $\mathcal{Y}'$ and $\mathcal{Y}''$ respectively. We need to check that the triangle

$$\begin{tikzcd}
D(\mathcal{Y}, T) \arrow[r, "L^{\lambda}_{T,T'}"] \arrow[dr, "L^{\lambda'}_{T,T''}"'] & D(\mathcal{Y}', T') \arrow[d, "L^{\lambda''}_{T',T''}"] \\
& D(\mathcal{Y}'', T'')
\end{tikzcd} \tag{5}$$

commutes. Using the Main Lemma three times, we obtain the diagram

$$
\begin{tikzcd}
D(\mathcal{Y},T) \arrow[r, "L^{\lambda}_{T,T''}"] \arrow[drr, "L^{\lambda'}_{T,T''}"] \arrow[dd, "\tilde{L}_T"'] & D(\mathcal{Y}',T') \arrow[dr, "L^{\lambda''}_{T',T''}"] \arrow[dd, "\tilde{L}_{T'}"] & \\
 & & D(\mathcal{Y}'',T'') \arrow[dd, "\tilde{L}_{T''}"'] \\
\mathcal{X} \arrow[r, "Id"] \arrow[drr, "Id"] & \mathcal{X} \arrow[dr, "Id"] & \\
 & & \mathcal{X},
\end{tikzcd}
$$

and see that the commutativity of (5) is equivalent to the commutativity of the tautological triangle

$$
\begin{tikzcd}
\mathcal{X} \arrow[r, "Id"] \arrow[dr, "Id"'] & \mathcal{X} \arrow[d, "Id"] \\
 & \mathcal{X}
\end{tikzcd}
$$

which is obvious.

5.2 The map from coverings to overgroups

We now show that there is a map

$$\underline{b} : \mathrm{Cov}(G(\mathcal{Y})) \to \mathrm{Over}(\Gamma)$$

Let $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a covering of complexes of groups, where $G'(\mathcal{Y}')$ is faithful and developable. For any maximal subtrees T and T' of $\mathcal{Y}$ and $\mathcal{Y}'$ respectively, let $\Lambda_{T,T'} : \pi_1(G(\mathcal{Y}),T) \to \pi_1(G'(\mathcal{Y}'),T')$ be the associated group monomorphism, and $L^{\lambda}_{T,T'} : D(\mathcal{Y},T) \to D(\mathcal{Y}',T')$ be the associated $\Lambda_{T,T'}$-equivariant isomorphism of scwols. Composition with the isomorphism $\tilde{L}_T^{-1}$ (see Proposition 27) yields an isomorphism of scwols

$$L_{\lambda,T'} = L^{\lambda}_{T,T'} \circ \tilde{L}_T^{-1} : \mathcal{X} \to D(\mathcal{Y}',T')$$

which is equivariant with respect to $\Lambda_{T,T'} \circ \Lambda_T^{-1} : \Gamma \to \pi_1(G'(\mathcal{Y}'),T')$. We set $\underline{b}(\lambda)$ to be the group

$$\underline{b}(\lambda) = L^{-1}_{\lambda,T'}(\pi_1(G'(\mathcal{Y}'),T'))L_{\lambda,T'}$$

which acts on $\mathcal{X}$. Since $G'(\mathcal{Y}')$ is faithful, $\pi_1(G'(\mathcal{Y}'),T')$ acts faithfully on $D(\mathcal{Y}',T')$. Hence we may identify $\underline{b}(\lambda)$ with a subgroup of $\mathrm{Aut}(K)$ which acts on $\mathcal{X}$. As $\Lambda_{T,T'}$ is injective, $\underline{b}(\lambda)$ is an overgroup of Γ.

Lemma 52 below shows that $\underline{b}$ is well-defined, that is, only depends on the isomorphism class of the covering λ.

Lemma 52 *Let $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ and $\lambda' : G(\mathcal{Y}) \to G''(\mathcal{Y}'')$ be isomorphic coverings of complexes of finite groups, with $G'(\mathcal{Y}')$ and $G''(\mathcal{Y}'')$ faithful and developable. Then $\underline{b}(\lambda) = \underline{b}(\lambda')$.*

PROOF. By definition, there exists an isomorphism $\lambda'' : G'(\mathcal{Y}') \to G''(\mathcal{Y}'')$ such that, for any choice of maximal trees, we have a commuting triangle

$$\begin{tikzcd} D(\mathcal{Y},T) \arrow[r, "L^{\lambda}_{T,T'}"] \arrow[dr, "L^{\lambda'}_{T,T''}"'] & D(\mathcal{Y}',T') \arrow[d, "L^{\lambda''}_{T',T''}"] \\ & D(\mathcal{Y}'',T'') \end{tikzcd}$$

and thus, composing with $\tilde{L}_T^{-1}$, a commuting triangle

$$\begin{tikzcd} \mathcal{X} \arrow[r, "L_{\lambda,T'}"] \arrow[dr, "L_{\lambda',T''}"'] & D(\mathcal{Y}',T') \arrow[d, "L^{\lambda''}_{T',T''}"] \\ & D(\mathcal{Y}'',T''). \end{tikzcd}$$

Since λ'' is an isomorphism, by Proposition 39 the group homomorphism $\Lambda_{T',T''} : \pi_1(G'(\mathcal{Y}'),T') \to \pi_1(G''(\mathcal{Y}''),T'')$ is an isomorphism. Thus, as $L^{\lambda''}_{T',T''}$ is $\Lambda_{T',T''}$-equivariant,

$$\begin{aligned} \underline{b}(\lambda') &= L^{-1}_{\lambda',T''}(\pi_1(G''(\mathcal{Y}''),T''))L^{-1}_{\lambda',T''} \\ &= L^{-1}_{\lambda,T'}(L^{\lambda''}_{T',T''})^{-1}(\pi_1(G''(\mathcal{Y}''),T''))L^{\lambda''}_{T',T''}L_{\lambda,T'} \\ &= L^{-1}_{\lambda,T'}(\pi_1(G'(\mathcal{Y}'),T'))L_{\lambda,T'} \\ &= \underline{b}(\lambda). \end{aligned}$$

Therefore $\underline{b}$ is well-defined.

5.3 Proof of Theorem 4

We now complete the proof of Theorem 4. Let $\underline{a} : \mathrm{Over}(\Gamma) \to \mathrm{Cov}(G(\mathcal{Y}))$ be as defined in Section 5.1 and $\underline{b} : \mathrm{Cov}(G(\mathcal{Y})) \to \mathrm{Over}(\Gamma)$ be as defined in Section 5.2.

Proposition 53 *The maps $\underline{a}$ and $\underline{b}$ are mutually inverse bijections.*

PROOF. We first prove that $\underline{b} \circ \underline{a} = 1$. For this, let Γ' be an overgroup of Γ acting without inversions, and let $\underline{a}(\Gamma') = \lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be an associated covering over a morphism of scwols $l : \mathcal{Y} \to \mathcal{Y}'$. By Lemma 50, we may assume that we can apply the Main Lemma to λ. For any maximal subtrees T and T' of $\mathcal{Y}$ and $\mathcal{Y}'$ respectively, we have then a commuting diagram of (equivariant) isomorphisms of scwols

$$\begin{tikzcd} D(\mathcal{Y}, T) \arrow[r, "L^{\lambda}_{T,T'}"] \arrow[d, "\tilde{L}_T"'] & D(\mathcal{Y}', T') \arrow[d, "\tilde{L}_{T'}"] \\ \mathcal{X} \arrow[r, "L=Id"'] & \mathcal{X}. \end{tikzcd}$$

Thus

$$\begin{aligned} \underline{b}(\lambda) &= L^{-1}_{\lambda,T'}(\pi_1(G'(\mathcal{Y}'), T'))L_{\lambda,T'} \\ &= (L^{\lambda}_{T,T'} \circ \tilde{L}^{-1}_T)^{-1}(\pi_1(G'(\mathcal{Y}'), T'))L^{\lambda}_{T,T'} \circ \tilde{L}^{-1}_T \\ &= \tilde{L}_{T'}(\pi_1(G'(\mathcal{Y}'), T'))\tilde{L}^{-1}_{T'} \\ &= \Gamma' \end{aligned}$$

since $\tilde{L}_{T'}$ is equivariant with respect to the isomorphism $\Lambda_{T'} : \pi_1(G'(\mathcal{Y}'), T') \to \Gamma'$. We conclude that $\underline{b}\ \underline{a}(\Gamma') = \Gamma'$.

We now prove that $\underline{a} \circ \underline{b} = 1$. Let $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a covering of a faithful developable complex of groups $G'(\mathcal{Y}')$ over a morphism of scwols $l : \mathcal{Y} \to \mathcal{Y}'$. Choose a vertex $\sigma_0 \in V(\mathcal{Y})$ and maximal trees T and T' in $\mathcal{Y}$ and $\mathcal{Y}'$ respectively. Without loss of generality, we identify $G'(\mathcal{Y}')$ with the complex of groups induced by the action of $\pi_1(G'(\mathcal{Y}'), T')$ on $D(\mathcal{Y}', T')$, using the isomorphism θ' defined in Lemma 41 above. By abuse of notation, we write λ for $\theta' \circ \lambda$. Let $\Gamma' = \underline{b}(\lambda)$.

Let $\mu = \underline{a}(\Gamma')$ be a covering $\mu : G(\mathcal{Y}) \to G''(\mathcal{Y}'')_{C''_\bullet}$ over a morphism of scwols $l' : \mathcal{Y} \to \mathcal{Y}''$, where $G''(\mathcal{Y}'')$ is a complex of groups induced by the action of Γ' on $\mathcal{X}$. By Lemma 50, we may assume that $\overline{\sigma_0} = \overline{l'(\sigma_0)}$ so that we can apply the Main Lemma to μ. We now show that λ and $\mu = \underline{a}\,\underline{b}(\lambda)$ are isomorphic coverings.

The map $\underline{b}$ induces a group isomorphism

$$\Lambda_{\underline{b}} : \pi_1(G'(\mathcal{Y}'), T') \to \underline{b}(\lambda)$$

with, for each $g' \in \pi_1(G'(\mathcal{Y}'), T')$ and each $\alpha \in \mathcal{X}$,

$$\Lambda_{\underline{b}}(g') \cdot \alpha = L^{-1}_{\lambda,T'}(g' \cdot L_{\lambda,T'}(\alpha)).$$

By construction, $L^{-1}_{\lambda,T'} : D(\mathcal{Y}',T') \to \mathcal{X}$ is $\Lambda_{\underline{b}}$-equivariant. Let $f : \mathcal{Y}' \to \mathcal{Y}''$ be the induced map of the quotient scwols

$$\mathcal{Y}' = \pi_1(G'(\mathcal{Y}'),T')\backslash D(\mathcal{Y}',T') \quad \text{and} \quad \mathcal{Y}'' = \Gamma'\backslash\mathcal{X}.$$

Since $\Lambda_{\underline{b}}$ and $L^{-1}_{\lambda,T'}$ are both isomorphisms, f is an isomorphism of scwols. We claim that the following diagram of morphisms of scwols commutes:

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "l"] \arrow[dr, "l'"'] & \mathcal{Y}' \arrow[d, "f"] \\ & \mathcal{Y}''. \end{tikzcd}$$

Let $\alpha \in \mathcal{Y}$. Then $\alpha = \Gamma\overline{\alpha}$ with $\overline{\alpha} \in \mathcal{X}$. We identify $l(\alpha) \in \mathcal{Y}'$ with the orbit $\pi_1(G'(\mathcal{Y}'),T')([1],l(\alpha)) = \pi_1(G'(\mathcal{Y}'),T')([u_{i(\alpha)}],l(\alpha))$. Then

$$f(l(\alpha)) = \Gamma' L^{-1}_{\lambda,T'}([u_{i(\alpha)}],l(\alpha)) = \Gamma' h_{i(\alpha)}\overline{\alpha} = \Gamma'\overline{\alpha} = l'(\alpha)$$

proving the claim.

We next choose elements $k_{\sigma'} \in \Gamma'$ such that, for each $\sigma' \in V(\mathcal{Y}')$,

$$k_{\sigma'}\, L^{-1}_{\lambda,T'}([1],\sigma') = \overline{f(\sigma')}.$$

We claim that $L^{-1}_{\lambda,T'}([1],l(\sigma_0)) = \overline{f(l(\sigma_0))}$. Now

$$L_{\lambda,T'}(\overline{f(l(\sigma_0))}) = L^{\lambda}_{T,T'} \circ \tilde{L}^{-1}_{T}(\overline{f(l(\sigma_0))}) = L^{\lambda}_{T,T'}([1],\sigma_0) = ([1],l(\sigma_0))$$

since $h_{i(f(l(\sigma_0)))} = 1$ and $u_{\sigma_0} = 1$, which proves the claim. Hence we may, and do, choose $k_{\sigma'_0} = 1$.

The elements $k_{\sigma'}$ then induce a morphism $\phi : G'(\mathcal{Y}') \to G''(\mathcal{Y}'')$ over f, given by $\phi_{\sigma'}(g') = k_{\sigma'}\Lambda_{\underline{b}}(g')k^{-1}_{\sigma'}$ for $g' \in G'_{\sigma'}$, and $\phi(a') = k_{t(a')}\Lambda_{\underline{b}}(a'^{+})k^{-1}_{i(a')}h^{-1}_{f(a')}$ for $a' \in E(\mathcal{Y}')$. Since $\Lambda_{\underline{b}}$ and f are isomorphisms, ϕ is an isomorphism of complexes of groups. Moreover, the following diagram commutes up to a homotopy from $\Lambda_{\underline{b}}\iota'_{T'}$ to $\phi''_1\phi$, given by the elements $\{k_{\sigma'}\}$:

$$\begin{tikzcd} G'(\mathcal{Y}') \arrow[r, "\iota'_{T'}"] \arrow[d, "\phi"'] & \pi_1(G'(\mathcal{Y}'),T') \arrow[d, "\Lambda_{\underline{b}}"'] \\ G''(\mathcal{Y}'') \arrow[r, "\phi''_1"] & \Gamma'. \end{tikzcd}$$

Hence, by Proposition 24, there is a $\Lambda_{\underline{b}}$-equivariant isomorphism of scwols

$$L_{\underline{b}} : D(\mathcal{Y}',T') \to D(\mathcal{Y}'',\phi''_1)$$

given explicitly by

$$([g'],\alpha') \mapsto ([\Lambda_{\underline{b}}(g')k^{-1}_{i(\alpha')}],f(\alpha')).$$

We now choose a maximal subtree T'' of $\mathcal{Y}''$ and compose $L_{\underline{b}}$ with the isomorphism $L_{T''}^{-1} : D(\mathcal{Y}'', \phi_1'') \to D(\mathcal{Y}'', T'')$ to obtain an isomorphism of scwols

$$L : D(\mathcal{Y}', T') \to D(\mathcal{Y}'', T'')$$

which is equivariant with respect to the composition of group isomorphisms

$$\Lambda_{T''}^{-1} \circ \Lambda_{\underline{b}} : \pi_1(G'(\mathcal{Y}'), T') \to \Gamma' \to \pi_1(G''(\mathcal{Y}''), T'').$$

Since $k_{\sigma_0'} = 1$ and $h_{f(\sigma_0')} = 1$,

$$L_{\underline{b}}([1], \sigma_0') = ([k_{\sigma_0'}], f(\sigma_0')) = ([h_{f(\sigma_0')}], f(\sigma_0')) = L_{T''}([1], f(\sigma_0'))$$

hence $L([1], \sigma_0') = ([1], f(\sigma_0'))$. We may thus apply the Corollary to the Main Lemma to L. We now have $L = L^{\lambda'}_{T',T''}$ for some morphism $\lambda' : G'(\mathcal{Y}') \to G''(\mathcal{Y}'')$. By Proposition 39, since L is an isomorphism of scwols which is equivariant with respect to an isomorphism of groups, λ' is an isomorphism of complexes of groups.

To complete the proof, it now suffices to show that the following diagram commutes:

$$\begin{tikzcd}
D(\mathcal{Y}, T) \arrow[r, "L^{\lambda}_{T,T'}"] \arrow[dr, "L^{\mu}_{T,T''}"'] & D(\mathcal{Y}', T') \arrow[d, "L=L^{\lambda'}_{T',T''}"] \\
& D(\mathcal{Y}'', T'').
\end{tikzcd}$$

By definition of L, it suffices to show that

$$L_{\underline{b}} \circ L^{\lambda}_{T,T'} = L_{T''} \circ L^{\mu}_{T,T''}.$$

Let $g \in \pi_1(G(\mathcal{Y}), T)$ and $\alpha \in \mathcal{Y}$. We write $u^{\lambda}_{i(\alpha)}$ for the element of $\pi_1(G'(\mathcal{Y}'), T')$ with respect to which $L^{\lambda}_{T,T'}$ is defined, and similarly for $u^{\mu}_{i(\alpha)} \in \pi_1(G''(\mathcal{Y}''), T'')$. Then

$$L_{\underline{b}} \circ L^{\lambda}_{T,T'}([g], \alpha) = \left(\left[\Lambda_{\underline{b}} \left\{ \Lambda_{T,T'}(g) u^{\lambda}_{i(\alpha)} \right\} k^{-1}_{i(l(\alpha))} \right], f(l(\alpha)) \right)$$

and

$$L_{T''} \circ L^{\mu}_{T,T''}([g], \alpha) = \left(\left[\Lambda_{T''} \left\{ \Lambda_{T,T''}(g) u^{\mu}_{i(\alpha)} \right\} h_{i(l'(\alpha))} \right], l'(\alpha) \right).$$

Since $f \circ l = l'$, it suffices to show that

$$\Lambda_{\underline{b}} \left\{ \Lambda_{T,T'}(g) u^{\lambda}_{i(\alpha)} \right\} k^{-1}_{i(l(\alpha))} \overline{f(l(\alpha))} = \Lambda_{T''} \left\{ \Lambda_{T,T''}(g) u^{\mu}_{i(\alpha)} \right\} h_{i(l'(\alpha))} \overline{l'(\alpha)}. \qquad (6)$$

By definition of the elements $k_{\sigma'}$, the left-hand side of (6) equals

$$\begin{aligned}
&\Lambda_{\underline{b}}\left\{\Lambda_{T,T'}(g)u^{\lambda}_{i(\alpha)}\right\} L^{-1}_{\lambda,T'}\left([1],l(\alpha)\right)\\
&\quad = L^{-1}_{\lambda,T'}\left(\Lambda_{T,T'}(g)u^{\lambda}_{i(\alpha)}\cdot([1],l(\alpha))\right) \quad \text{since } L^{-1}_{\lambda,T'} \text{ is } \Lambda_{\underline{b}}\text{-equivariant}\\
&\quad = L^{-1}_{\lambda,T'}\left(\Lambda_{T,T'}(g)\cdot([u^{\lambda}_{i(\alpha)}],l(\alpha))\right)\\
&\quad = L^{-1}_{\lambda,T'}\left(\Lambda_{T,T'}(g)\cdot L^{\lambda}_{T,T'}([1],\alpha)\right)\\
&\quad = L^{-1}_{\lambda,T'}\circ L^{\lambda}_{T,T'}([g],\alpha) \quad \text{since } L^{\lambda}_{T,T'} \text{ is } \Lambda_{T,T'}\text{-equivariant}\\
&\quad = \tilde{L}_T([g],\alpha) \quad \text{by definition of } L_{\lambda,T'}.
\end{aligned}$$

On the right-hand side of (6), we have, by definition of $\tilde{L}_{T''}$,

$$\begin{aligned}
&\Lambda_{T''}\left\{\Lambda_{T,T''}(g)u^{\mu}_{i(\alpha)}\right\} \tilde{L}_{T''}([1],l'(\alpha))\\
&\quad = \tilde{L}_{T''}\left(\Lambda_{T,T''}(g)u^{\mu}_{i(\alpha)}\cdot([1],l'(\alpha))\right) \quad \text{since } \tilde{L}_{T''} \text{ is } \Lambda_{T''}\text{-equivariant}\\
&\quad = \tilde{L}_{T''}\left(\Lambda_{T,T''}(g)\cdot([u^{\mu}_{i(\alpha)}],l'(\alpha))\right)\\
&\quad = \tilde{L}_{T''}\left(\Lambda_{T,T''}(g)\cdot L^{\mu}_{T,T''}([1],\alpha)\right)\\
&\quad = \tilde{L}_{T''}\circ L^{\mu}_{T,T''}([g],\alpha) \quad \text{since } L^{\mu}_{T,T''} \text{ is } \Lambda_{T,T''}\text{-equivariant.}
\end{aligned}$$

But by the Main Lemma applied to μ, we have a commuting square

$$\begin{tikzcd}
D(\mathcal{Y},T) \arrow[r, "L^{\mu}_{T,T''}"] \arrow[d, "\tilde{L}_T"'] & D(\mathcal{Y}'',T'') \arrow[d, "\tilde{L}_{T''}"] \\
\mathcal{X} \arrow[r, "Id"'] & \mathcal{X}
\end{tikzcd}$$

hence equation (6) holds.

We conclude by establishing a bijection between n–sheeted coverings and overlattices of index n.

Corollary 54 *Let K be a simply connected, locally finite polyhedral complex, and let Γ be a cocompact lattice in* $\mathrm{Aut}(K)$ *(acting without inversions) which induces a complex of groups $G(\mathcal{Y})$. Then there is a bijection between the set of overlattices of Γ of index n (acting without inversions) and the set of isomorphism classes of n–sheeted coverings of faithful developable complexes of groups by $G(\mathcal{Y})$.*

PROOF. By the definition of n–sheeted covering, the bijection of Theorem 4 sends an isomorphism class of finite-sheeted coverings to an overgroup containing Γ with finite index.

Since Γ is cocompact, the quotient scwol $\mathcal{Y}$ is finite and the local groups G_σ of $G(\mathcal{Y})$ are finite groups. Let $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a finite-sheeted covering, where $G'(\mathcal{Y}')$ is a faithful, developable complex of groups. Then $\mathcal{Y}'$ is finite by Lemma 45, and the local groups $G'_{\sigma'}$ are finite since λ is finite-sheeted. It follows that the overgroup $\underline{b}(\lambda)$ is a cocompact lattice acting without inversions on K.

It remains to show that the bijection $\underline{a}$ sends an overlattice Γ' of index n to an n'-sheeted covering, with $n = n'$. Let $\lambda = \underline{a}(\Gamma') : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a covering associated to Γ', over the morphism of quotient scwols $l : \Gamma\backslash\mathcal{X} \to \Gamma'\backslash\mathcal{X}$. Then

$$
\begin{aligned}
n = [\Gamma' : \Gamma] &= \frac{\mathrm{Vol}(\Gamma\backslash\backslash V(\mathcal{X}))}{\mathrm{Vol}(\Gamma'\backslash\backslash V(\mathcal{X}))} = \frac{\sum_{\sigma \in V(\mathcal{Y})} \frac{1}{|G_\sigma|}}{\sum_{\sigma' \in V(\mathcal{Y}')} \frac{1}{|G'_{\sigma'}|}} \\
&= \frac{\sum_{\sigma' \in V(\mathcal{Y}')} \sum_{\sigma \in l^{-1}(\sigma')} \frac{1}{|G_\sigma|}}{\sum_{\sigma' \in V(\mathcal{Y}')} \frac{1}{|G'_{\sigma'}|}} = \frac{\sum_{\sigma' \in V(\mathcal{Y}')} \frac{n'}{|G'_{\sigma'}|}}{\sum_{\sigma' \in V(\mathcal{Y}')} \frac{1}{|G'_{\sigma'}|}} = n'
\end{aligned}
$$

as required.

We remark that we can define isomorphism between two coverings $\lambda : G'(\mathcal{Y}') \to G(\mathcal{Y})$ and $\lambda : G''(\mathcal{Y}'') \to G(\mathcal{Y})$ analogous to Definition 48 so that there is a bijection between the set of subgroups of Γ of index n and the set of isomorphism classes of n–sheeted coverings of $G(\mathcal{Y})$ by faithful developable complexes of groups. Since the proof is similar to that of Corollary 54, we omit it. Note that the developability comes free by Lemma 43.

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