Non-positivity of the semigroup generated by the Dirichlet-to-Neumann operator

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The Dirichlet-to-Neumann operator

Assume

- $\Omega \subseteq \mathbb{R}^N$ smooth bounded open domain
- $\lambda \in \mathbb{R}$

If $\lambda \notin \sigma(-\Delta)$, then for every $\varphi \in H^{1/2}(\partial\Omega)$

$$\begin{aligned} \Delta u + \lambda u &= 0 \quad \text{in } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega \end{aligned}$$

has a unique solution $u \in H^1(\Omega)$.

Definition

The Dirichlet-to-Neumann operator is defined by

$$D_\lambda \varphi := \frac{\partial u}{\partial \nu} \in H^{-1/2}(\Omega)$$

where ν is the outer unit normal to $\partial\Omega$

Some known facts

Let $0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots$ be the distinct eigenvalues of

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

If λ is not one of these eigenvalues, then

- $D_\lambda : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is bounded;
- $-D_\lambda$ generates an analytic semigroup on $L^2(\partial\Omega)$.

Moreover,

- if $\lambda < \lambda_1$, then e^{-tD_λ} is a positive irreducible semigroup; see Arendt and Mazzeo (2012)

Aim of Talk

Investigate positivity/non-positivity of e^{-tD_λ} for $\lambda > \lambda_1$.

Possible conjecture

In many cases, crossing a principal eigenvalue will result in loss of positivity and/or maximum principles.

First Conjecture

e^{-tD_λ} is not positive for all $\lambda > \lambda_1$;

This conjecture is disproved by the simplest example, namely $\Omega = (0, L) \subseteq \mathbb{R}$ an interval.

The Dirichlet-to-Neumann semigroup on $(0, L)$

Solving

$$u'' + \lambda u = 0 \qquad u(0) = a,\ u(L) = b$$

for $\lambda > \lambda_1 = (\pi/L)^2$ gives

$$u(x) = a\frac{\sin\sqrt{\lambda}(L-x)}{\sin\sqrt{\lambda}L} + b\frac{\sin\sqrt{\lambda}x}{\sin\sqrt{\lambda}L}.$$

and therefore

$$D_\lambda \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -u'(0) \\ u'(L) \end{bmatrix} = \frac{\sqrt{\lambda}}{\sin\sqrt{\lambda}L} \begin{bmatrix} \cos\sqrt{\lambda}L & -1 \\ -1 & \cos\sqrt{\lambda}L \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

- Hence we have the matrix representation

  $$D_\lambda = \begin{bmatrix} \alpha(\lambda) & -\beta(\lambda) \\ -\beta(\lambda) & \alpha(\lambda) \end{bmatrix},$$

  where

  $$\alpha(\lambda) := \frac{\sqrt{\lambda}\cos\sqrt{\lambda}L}{\sin\sqrt{\lambda}L} \qquad \text{and} \qquad \beta(\lambda) := \frac{\sqrt{\lambda}}{\sin\sqrt{\lambda}L}.$$

- The Dirichlet-to-Neumann semigroup is given by

  $$e^{-tD_\lambda} = e^{-t\alpha(\lambda)} \begin{bmatrix} \cosh\beta(\lambda)t & \sinh\beta(\lambda)t \\ \sinh\beta(\lambda)t & \cosh\beta(\lambda)t \end{bmatrix},$$

- Hence e^{-tD_λ} is positive if and only if

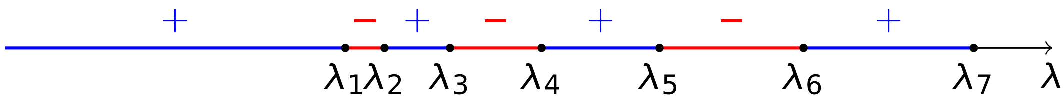
  $$\sinh\beta(\lambda)t \geq 0 \text{ for all } t \geq 0 \quad \Leftrightarrow \quad \sin\sqrt{\lambda}L > 0.$$

Conclusion

e^{-tD_λ} is positive if and only if $\lambda < \left(\frac{\pi}{L}\right)^2$ or

$$\left(\frac{2k\pi}{L}\right)^2 < \lambda < \left(\frac{(2k+1)\pi}{L}\right)^2, \quad k \geq 1.$$

That is, positivity and non-positivity of e^{-tD_λ} alternate at each eigenvalue:



The spectrum of D_λ

- A necessary condition for e^{-D_λ} to be positive is that the minimal eigenvalue of D_λ has a positive eigenvector.
- We have

$$\begin{bmatrix} \alpha(\lambda) & -\beta(\lambda) \\ -\beta(\lambda) & \alpha(\lambda) \end{bmatrix} \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} = \big(\alpha(\lambda) \mp \beta(\lambda)\big) \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix}$$

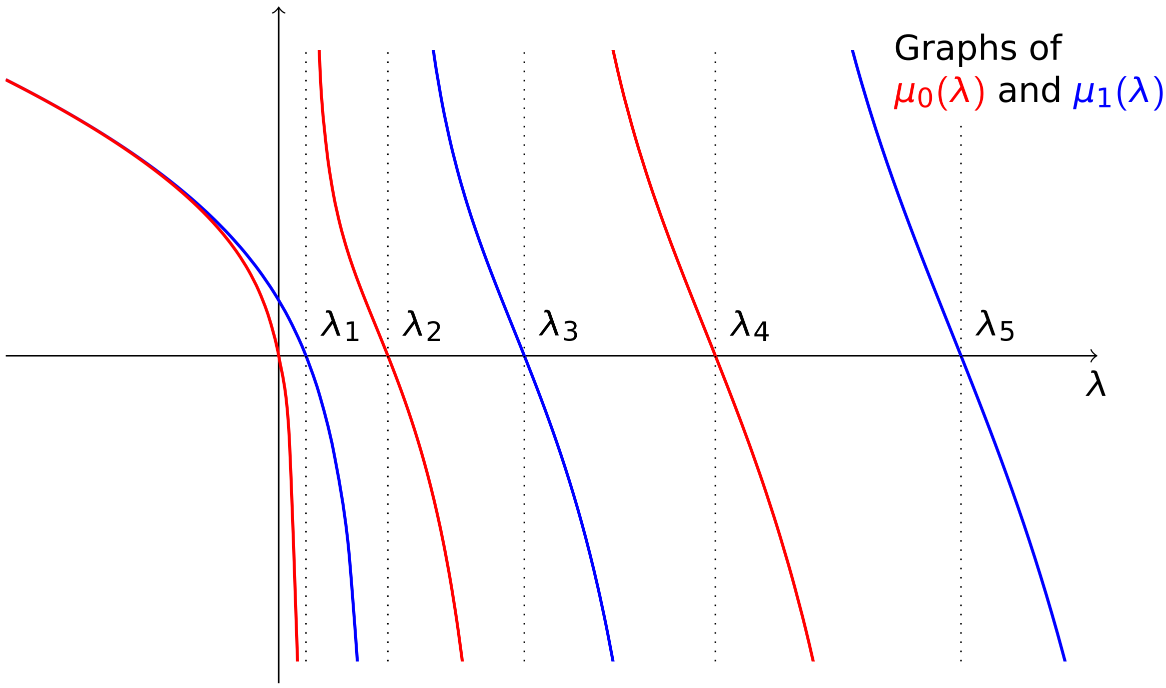
- Hence the eigenvalues/eigenvectors are

$$\mu_0(\lambda) = \alpha(\lambda) - \beta(\lambda) = -\sqrt{\lambda}\tan\frac{\sqrt{\lambda}L}{2} \quad \text{e-vect} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \gg 0;$$

$$\mu_1(\lambda) = \alpha(\lambda) - \beta(\lambda) = \sqrt{\lambda}\cot\frac{\sqrt{\lambda}L}{2} \quad \text{e-vect} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \not> 0.$$

Hence a necessary condition for positivity of e^{-tD_λ} is

$$\mu_0(\lambda) < \mu_1(\lambda)$$



Possible modifed conjectures

Second conjectures

- Positivity and non-positivity of e^{-tD_λ} alternate at each eigenvalue λ_k, possibly counting multiplicity.
- If e^{-tD_λ} is positive for some $\lambda \in (\lambda_k, \lambda_{k+1})$, then it is positive for all $\lambda \in (\lambda_k, \lambda_{k+1})$.
- If $\lambda > \lambda_1$, then e^{-tD_λ} is not positive in higher dimensions.

We show that all these conjectures are disproved by the example of the disc in $\mathbb{R}^2$.

The Dirichlet-to-Neumann operator on the disc

- Given $\varphi \in H^{1/2}(\partial B)$ solve

$$\Delta u + \lambda u = 0 \quad \text{in } B, \qquad u = \varphi \quad \text{on } \partial B. \qquad \text{(BVP)}$$

- We compute u for an orthogonal basis on $L^2(\partial B)$:

$$\varphi_k = e^{ik\theta}, \qquad k \in \mathbb{Z}.$$

- The solution of (BVP) is

$$u_k(r,\theta) = \frac{J_k(\sqrt{\lambda} r)}{J_k(\sqrt{\lambda})} e^{ik\theta},$$

- Hence, for $k \in \mathbb{Z}$,

$$D_\lambda e^{ik\theta} = \frac{\partial u_k}{\partial \nu} = \frac{\partial}{\partial r} \frac{J_k(\sqrt{\lambda} r)}{J_k(\sqrt{\lambda})} e^{ik\theta} \Big|_{r=1} = \frac{\sqrt{\lambda} J_k'(\sqrt{\lambda})}{J_k(\sqrt{\lambda})} e^{ik\theta}$$

Note that $e^{ik\theta}$ is an eigenfunction of D_λ.

As $J_{-k}(s) = (-1)^k J_k(s)$ the eigenvalue

$$\mu_k(\lambda) := \frac{\sqrt{\lambda} J_k'(\sqrt{\lambda})}{J_k(\sqrt{\lambda})}, \qquad k = 0, 1, 2, \ldots$$

has eigenfunctions $e^{\pm ik\theta}$.

Operator and semigroup on $L^2(\partial\Omega)$

If $\varphi = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta} \in H^{1/2}(\partial B)$, then

$$D_\lambda \varphi = \sum_{k=-\infty}^{\infty} c_k \mu_{|k|}(\lambda) e^{ik\theta}$$

$$e^{-tD_\lambda} \varphi = \sum_{k=-\infty}^{\infty} c_k e^{-t\mu_{|k|}(\lambda)} e^{ik\theta}$$

Spectrum of D_λ on the disc

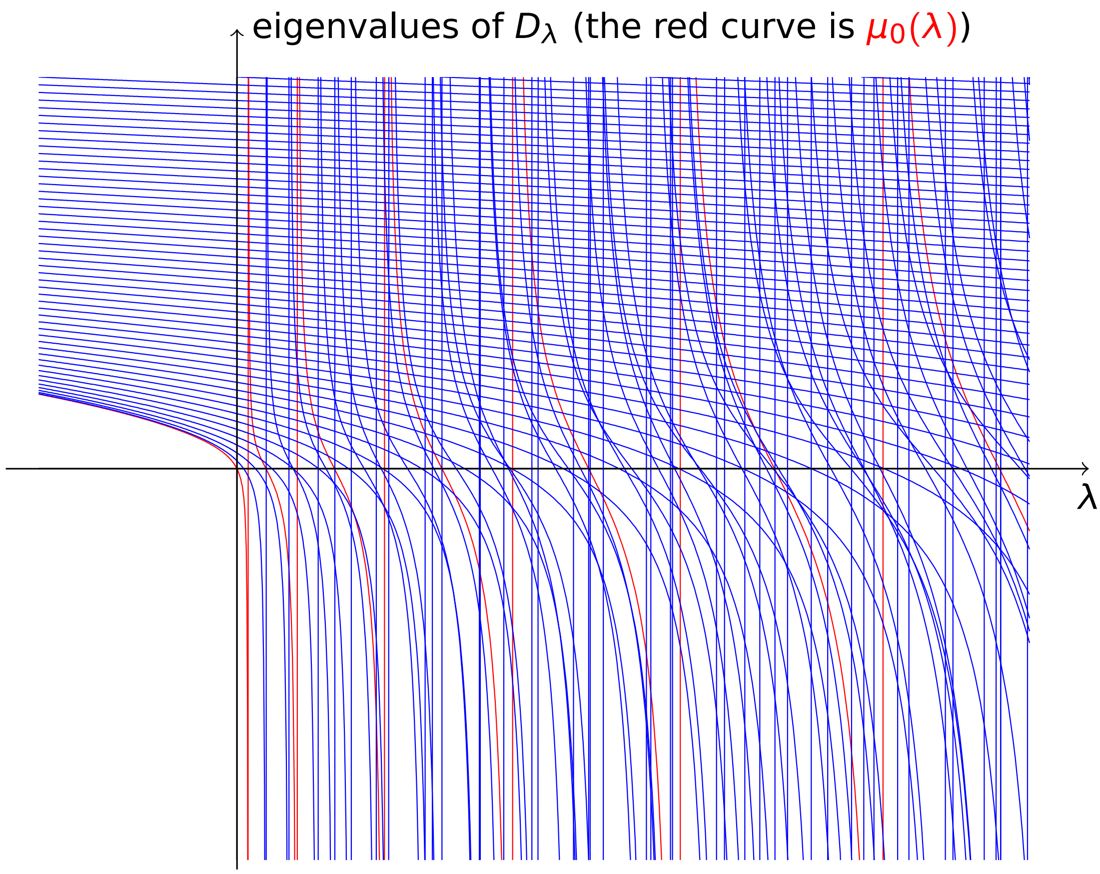
- The eigenspace to the eigenvalues $\mu_k(\lambda)$, $k = 0, 1, 2, \ldots,$ is spanned by

$$\cos k\theta, \quad \sin k\theta.$$

- $\mu_0(\lambda)$ is the only eigenvalue having a positive eigenfunction.
- Hence a necessary condition for e^{-tD_λ} to be positive is that

$$\mu_0(\lambda) < \mu_k(\lambda) \qquad \text{for all } k > 0. \tag{1}$$

- We shall show that (1) is not sufficient for e^{-tD_λ} to be positive.



The Dirichlet eigenvalues λ_n "jumble" the order of $\mu_k(\lambda)$

Where can e^{-D_λ} be positive?

- From the graph we see that e^{-tD_λ} can only be positive in a left neighbourhood of some of the Dirichlet eigenvalues, namely where

$$\lim_{\lambda \to \lambda_k -} \mu_0(\lambda) = -\infty.$$

- Recall that

$$\mu_0(\lambda) = \frac{\sqrt{\lambda} J_0'(\sqrt{\lambda})}{J_0(\sqrt{\lambda})}.$$

- These are the Dirichlet eigenvalues of $-\Delta$ determined by the zeros of J_0.

Fourier series of non-negative functions

- Let

$$\varphi = \sum_{-\infty}^{\infty} c_k e^{ik\theta} \geq 0.$$

Then $c_{-k} = \bar{c}_k$ and (c_k) is positive definite.

- Indeed, if $\xi_k \in \mathbb{C}$, then

$$\sum_{j,k=1}^{n} c_{k-j}\xi_k\bar{\xi}_j = \sum_{j,k=1}^{n} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ij\theta}e^{ik\theta}\xi_k\bar{\xi}_j\varphi(\theta)\,d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{k=1}^{n} e^{-ik\theta}\xi_k\right|^2\varphi(\theta)\,d\theta \geq 0.$$

- In particular $c_0 \geq |c_n|$ for all $n \in \mathbb{Z}$. Choose $\xi_0 = 1$, $\xi_n = \alpha$ with $|\alpha| = 1$ so that $\alpha c_n = -|c_n|$ and $\xi_j = 0$:

$$\sum_{j,k=1}^{n} c_{k-j}\xi_k\bar{\xi}_j = 2c_0 + \alpha c_{0-n} + \bar{\alpha} c_{n-0} = 2(c_0 - |c_n|) \geq 0.$$

Eventual positivity & irreducibility

Theorem

Let $\mu_0(\lambda) < \mu_k(\lambda)$ for all $k \geq 1$.

- *There exists $T > 0$ such that the operator e^{-tD_λ} is positive and irreducible for all $t \geq T$.*
- *It is possible that e^{-tD_λ} is not positive for all $t > 0$.*

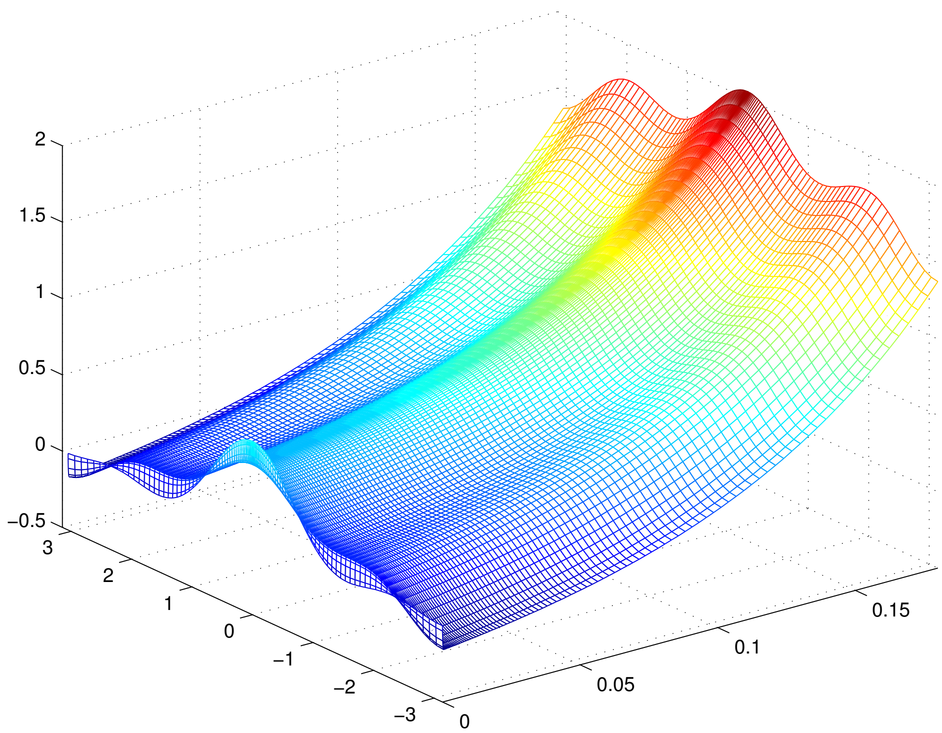
If $\varphi = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta} \geq 0$, then $c_0 \geq |c_k|$ and

$$e^{-tD_\lambda}\varphi = \sum_{k=-\infty}^{\infty} c_k e^{-t\mu_{|k|}(\lambda)} e^{ik\theta} \geq c_0 e^{-t\mu_0(\lambda)} - 2c_0 \sum_{k=1}^{\infty} e^{-t\mu_k(\lambda)}$$

$$= c_0 e^{-t\mu_0(\lambda)} \left(1 - 2\sum_{k=1}^{\infty} e^{-t(\mu_k(\lambda)-\mu_0(\lambda))}\right) > 0$$

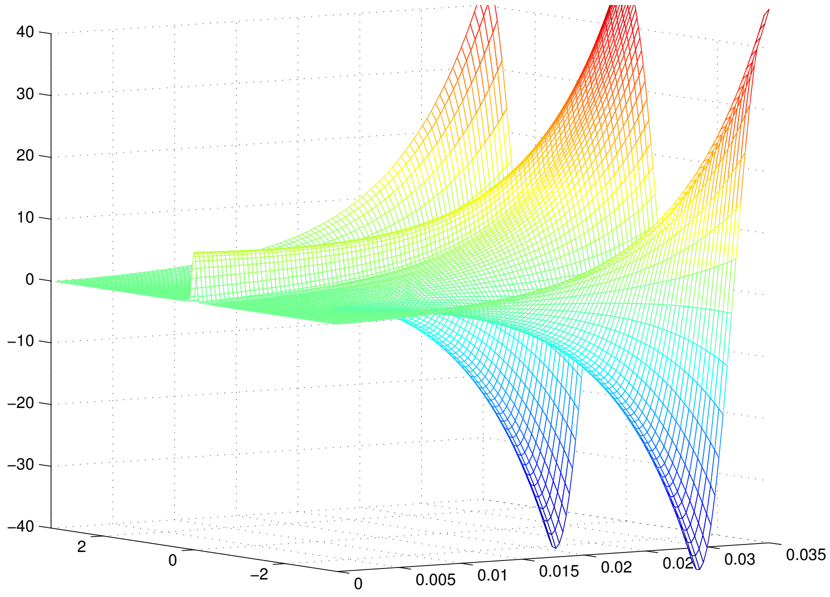
for t large enough, independently of $\varphi \geq 0$.

Non-positivity if $\mu_0(\lambda) < \mu_k(\lambda)$ for all $k \geq 1$



The solution may dip below zero for some $\lambda \in (\lambda_3, \lambda_4)$.

Non-positivity if $\mu_m(\lambda) < \mu_0(\lambda)$ for some $m > 0$



An oscillating term dominates the series

$$e^{-tD_\lambda}\varphi = \sum_{k=-\infty}^{\infty} c_k e^{-t\mu_{|k|}(\lambda)} e^{ik\theta}.$$

Positivity of e^{-tD_λ}

Theorem

For each Dirichlet eigenvalue λ_ℓ such that $J_0(\sqrt{\lambda_\ell}) = 0$ there exists $\beta < \lambda_\ell$ such that e^{-tD_λ} is a positive semigroup for all $\lambda \in [\beta, \lambda_\ell)$.

- Write

$$e^{-tD_\lambda}\varphi = \sum_{-\infty}^{\infty} c_k e^{-t\mu_{|k|}} e^{ik\theta}$$

$$= G_{\lambda,t} * \varphi := \int_{-\pi}^{\pi} G_{\lambda,t}(\theta - \cdot)\varphi(\theta)\, d\theta.$$

- $G_{\lambda,t}$ is the "heat kernel" of e^{-tD_λ} given by

$$G_{\lambda,t}(\theta) := \sum_{k=-\infty}^{\infty} e^{-t\mu_{|k|}} e^{ik\theta} \qquad t > 0.$$

Positivity of e^{-tD_λ} ...

- Show that $G_{\lambda,t}(\theta) \geq 0$ for all $t > 0$ for λ in some interval $[\beta, \lambda_\ell)$.
- $G_{\lambda,t}(\theta) \geq 0$ if and only if the sequence

$$e^{-t\mu_{|k|}(\lambda)}$$

  of Fourier coefficients is positive definite.
- Positive definiteness is hard to check but there is a sufficient condition, Polya's criterion:
  - $c_k \to 0$
  - $k \mapsto c_k$ is convex, that is, $c_{k-1} + c_{k+1} - 2c_k \geq 0$
- Express the Fourier series in terms of the Féjer kernels $K_n(\theta) \geq 0$ in the form

$$\sum_{n=1}^{\infty} n(c_{k-1} + c_{k+1} - 2c_k)K_{n-1}(\theta) \geq 0.$$

Positivity of e^{-tD_λ} ...

Polya's criterion is only a sufficient condition.

However the formula is still valid if the sequence of Fourier coefficients is eventually convex.

Proposition

$$G_{\lambda,t}(\theta) = \sum_{n=1}^{\infty} nb_n(\lambda, t)K_{n-1}(\theta),$$

where

$$b_n(\lambda, t) := e^{-t\mu_{n+1}(\lambda)} + e^{-t\mu_{n-1}(\lambda)} - 2e^{-t\mu_n(\lambda)}.$$

Moreover, $b_n(\lambda, t) \geq 0$ for all $n > \sqrt{\lambda}$ and all $t > 0$.

Positivity of e^{-tD_λ} ...

- If $j_{k,\ell}$ are the positive zeros of J_k, then

$$\mu_k(\lambda) = \frac{\sqrt{\lambda} J_k'(\sqrt(\lambda))}{J_k(\lambda)} = \sum_{\ell=1}^{\infty} \frac{2\lambda}{j_{k,\ell}^2 - \lambda};$$

- It is shown in Elbert and Laforgia (1984) that

$$k \to j_{k,\ell}^2 \qquad \text{is concave.}$$

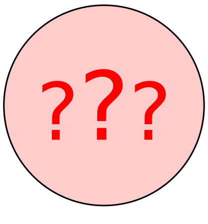
- Hence $k \mapsto e^{-\mu_k(\lambda)}$ is eventually convex.
- This means almost all terms in the series

$$G_{\lambda,t}(\theta) = \sum_{n=1}^{\infty} n b_n(\lambda, t) K_{n-1}(\theta)$$

  are positive.
- If $\mu_0(\lambda) \ll 0$, then the sum of the finitely many terms is postitive for all $t > 0$.

Open Questions

What happens on more general domains?



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