Discrete Painlevé equations and special functions: an overview

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Main References:

[KMNOY2003]

K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada: ${}_{10}E_9$ solution to the elliptic Painlevé equation, J. Phys. A **36** (2003), L263 – L272.

[KMNOY2006]

K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada: Point configurations, Cremona transformations and the elliptic difference Painlevé equation, in *Théories asymptotiques et équations de Painlevé*, pp.169 – 198, Sémin. Congr. **14**, Soc. Math. France, Paris, 2006.

Six Painlevé equations

$$P_{\mathrm{I}}: \quad y'' = 6y^2 + t \qquad\qquad P_{\mathrm{II}}: \quad y'' = 2y^3 + ty + \alpha$$

$$P_{\mathrm{III}}: \quad y'' = \frac{1}{y}(y')^2 - \frac{1}{t}y' + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$$

$$P_{\mathrm{IV}}: \quad y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}$$

$$P_{\mathrm{V}}: \quad y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma}{t}y + \delta\frac{y(y+1)}{y-1}$$

$$P_{\mathrm{VI}}: \quad y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)(y')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)$$

Consider a second-order nonlinear ordinary differential equation $y'' = R(t; y, y')$ for $y = y(t)$, where $R(t; y, \eta)$ is a rational function in (t, y, η), and $' = d/dt$. Such an equation is said to have the *Painlevé property* if any solution has no movable singular point except for poles.

Any rational ordinary differential equation of second order with the Painlevé property is reduced to one of the six Painlevé equations, $P_{\mathrm{I}}, \ldots, P_{\mathrm{VI}}$ unless it can be integrated algebraically, or transformed into a simpler equation such as a linear equation or the differential equation of elliptic functions.

Characteristic features of Painlevé equations

- **Degeneration diagram:**

$$\begin{array}{ccccccccc} P_{\rm VI} & \rightarrow & P_{\rm V} & \rightarrow & P_{\rm III} & & & & \\ & & & \searrow & & \searrow & & & \\ & & & & P_{\rm IV} & \rightarrow & P_{\rm II} & \rightarrow & P_{\rm I} \end{array}$$

- **Affine Weyl group symmetry:**
  (Bäcklund transformations)

$$\begin{array}{ccccccccc} D_4^{(1)} & \rightarrow & A_3^{(1)} & \rightarrow & (2A_1)^{(1)} & & & & \\ & & & \searrow & & \searrow & & & \\ & & & & A_2^{(1)} & \rightarrow & A_1^{(1)} & \rightarrow & \cdot \end{array}$$

- **Hypergeometric solutions:**

$$\begin{array}{ccccccccc} \text{Gauss} & \rightarrow & \text{Kummer} & \rightarrow & \text{Bessel} & & & & \\ & & & \searrow & & \searrow & & & \\ & & & & \text{Hermite} & \rightarrow & \text{Airy} & \rightarrow & \cdot \end{array}$$

The parameter space of $P_{\rm J}$ ($\rm J = II, \ldots, VI$) can be identified with the Cartan subalgebra of a semisimple Lie algebra. The natural action of the corresponding affine Weyl group on the parameter space can be lifted to a group of Bäcklund transformations for $P_{\rm J}$. (Okamoto 1970s, ...)

Along each wall of the affine Weyl group in the parameter space, there arise special solutions which can be expressed in terms of (confluent) hypergeometric functions. Typically, algebraic solutions arise as fixed points of some diagram automorphism of the affine Weyl group. (Umemura 1980s, ...)

An example: $P_{\rm IV}$

- **Symmetric form of $P_{\rm IV}$**

$$P_{\rm IV}: y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2-\alpha)y + \frac{\beta}{y} \quad \Longleftrightarrow \quad \begin{cases} f_0' = f_0(f_1 - f_2) + \alpha_0 \\ f_1' = f_1(f_2 - f_0) + \alpha_1 \\ f_2' = f_2(f_0 - f_1) + \alpha_2 \end{cases} \quad f_0 + f_1 + f_2 = t.$$

Here α_j $(j = 0, 1, 2)$ are complex parameters with $\alpha_0 + \alpha_1 + \alpha_2 = 1$.
The differential equation for $y(t) = -f_1(\sqrt{2}t)/\sqrt{2}$ coincides with $P_{\rm IV}$ with $\alpha = \alpha_0 - \alpha_2$, $\beta = -2\alpha_1^2$.

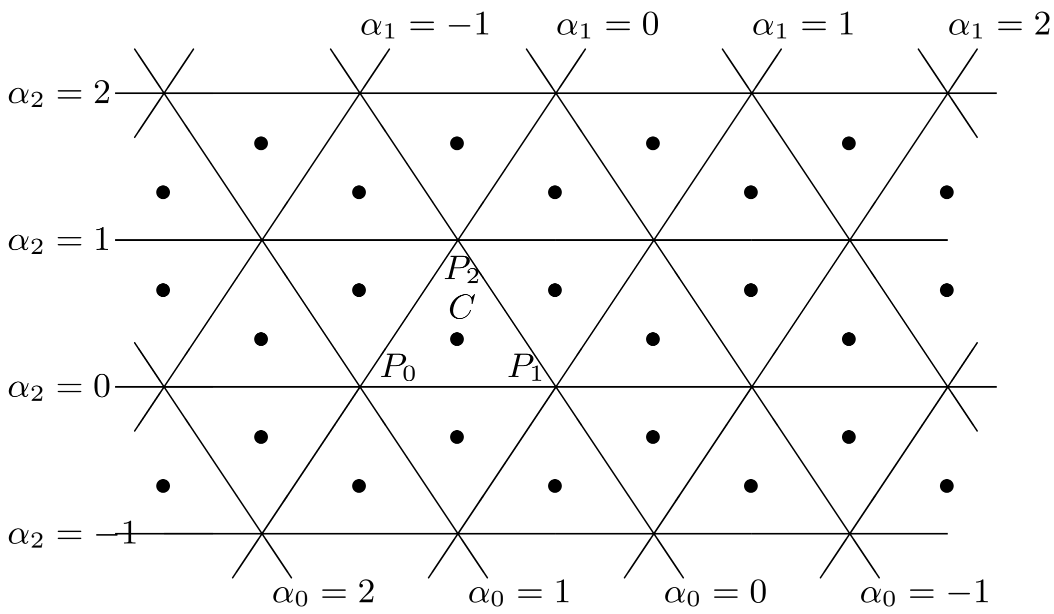
- **Affine Weyl group symmetry**

Consider the differential field $\mathcal{K} = \mathbb{C}(\alpha_0, \alpha_1, \alpha_2; f_0, f_1, f_2)$ representing the symmetric form of $P_{\rm IV}$. We regard the indices for α_j and f_j as elements of $\mathbb{Z}/3\mathbb{Z}$. This differential field $\mathcal{K}$ has fundamental differential automorphisms (Bäcklund transformations) s_0, s_1, s_2 and π:

$$\begin{aligned} &s_i(\alpha_i) = -\alpha_i, && s_i(\alpha_{i+1}) = \alpha_{i+1} + \alpha_i, && s_i(\alpha_{i-1}) = \alpha_{i-1} + \alpha_i, \\ &s_i(f_i) = f_i, && s_i(f_{i+1}) = f_{i+1} + \frac{\alpha_i}{f_i}, && s_i(f_{i-1}) = f_{i-1} - \frac{\alpha_i}{f_i}. \\ &\pi(\alpha_i) = \alpha_{i+1}, && \pi(f_i) = f_{i+1} \end{aligned}$$

These transformations generate the (extended) affine Weyl group $\widetilde{W} = \langle s_0, s_1, s_2, \pi \rangle$ of type $A_2^{(1)}$:

$$\begin{aligned} &s_i^2 = 1 \quad (i = 0, 1, 2); \quad (s_0 s_1)^3 = (s_1 s_2)^3 = (s_2 s_0)^3 = 1; \\ &\pi^3 = 1; \quad \pi s_i = s_{i+1}\pi \quad (i = 0, 1, 2). \end{aligned}$$



- **Hypergeometric solutions**

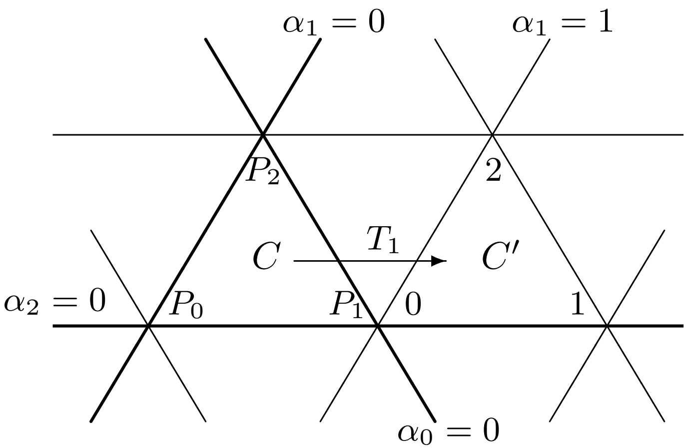
When $\alpha_0 = 0$, by setting $f_0 = 0$ we obtain the first order equation $f_1' = f_1 f_2 + \alpha_1 = f_1(t - f_1) + \alpha_1$. This Riccati equation is linearized to the Hermite equation $u'' - tu' - \alpha_1 u = 0$ by the transformation of variables $f_1 = u'/u$. Along each wall (reflection hyperplane) $\alpha_j = n$ ($j = 0, 1, 2; n \in \mathbb{Z}$), there exists a one-parameter family of solutions which are expressed by determinants of Hermite functions. They are obtained by Bäcklund transformation from the seed solutions along $\alpha_0 = 0$.

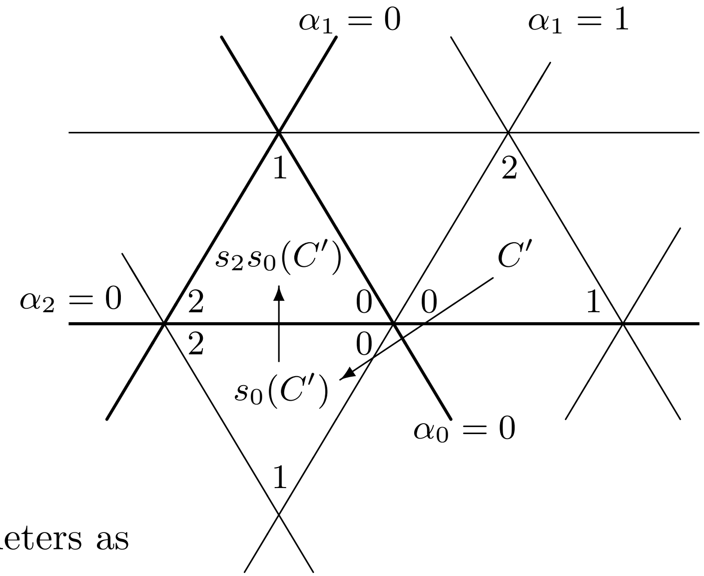
- **Algebraic solutions**

At the barycenter $(\alpha_0, \alpha_1, \alpha_2) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ of the fundamental alcove C, there is a rational solution $(f_0, f_1, f_2) = (\frac{t}{3}, \frac{t}{3}, \frac{t}{3})$, obtained as the fixed point of π. By Bäcklund transformations, we obtain a rational solution at the barycenter of each triangle (expressed by ratios of Okamoto polynomials).

• Discrete Painlevé equation as Schlesinger transformation

Consider the parallel translation T_1 by the vector $\overrightarrow{P_0P_1}$. This transformation T_1 belongs to the extended affine Weyl group $\widetilde{W} = \langle s_0, s_1, s_2, \pi \rangle$, and is expressed as $T_1 = s_0 s_2 \pi = \pi s_2 s_1$.





This element $T_1 = \pi s_2 s_1 \in \widetilde{W}$ transforms the parameters as

$$T_1(\alpha_0) = \alpha_0 + 1, \quad T_1(\alpha_1) = \alpha_1 - 1, \quad T_1(\alpha_2) = \alpha_2.$$

The action of T_1 on the dependent variables f_j $(j = 0, 1, 2)$ is computed as

$$T_1(f_0) = f_1 + \frac{\alpha_0}{f_0} - \frac{\alpha_2 + \alpha_0}{f_1 - \frac{\alpha_0}{f_0}} = \frac{f_0^2 f_1 f_2 - \alpha_0 f_0 f_1 + \alpha_0 f_2 f_0 - (\alpha_0 + \alpha_2) f_0^2 - \alpha_0^2}{f_0(f_0 f_2 - \alpha_0)},$$

$$T_1(f_1) = f_2 - \frac{\alpha_0}{f_0} = \frac{f_0 f_2 - \alpha_0}{f_0},$$

$$T_1(f_2) = f_0 + \frac{\alpha_0 + \alpha_2}{f_2 - \frac{\alpha_0}{f_0}} = \frac{f_0(f_0 f_2 + \alpha_2)}{f_0 f_2 - \alpha_0}.$$

This can be regarded as a version of the *discrete Painlevé equation* dP_{II}.

Note that

$$T_1(f_1) = f_2 - \frac{\alpha_0}{f_0} = t - f_0 - f_1 - \frac{\alpha_0}{f_0}, \quad T_1^{-1}(f_0) = f_2 + \frac{\alpha_1}{f_1} = t - f_0 - f_1 + \frac{\alpha_1}{f_1}.$$

By setting $x_n = T_1^n(f_1), \quad y_n = T_1^n(f_0) \quad (n \in \mathbb{Z})$, we obtain

$$x_n + x_{n+1} = t - y_n - \frac{\alpha_0 + n}{y_n}, \quad y_{n-1} + y_n = t - x_n + \frac{\alpha_1 - n}{x_n}.$$

Here n is the independent variable and t, α_0, α_1 are the parameters. This equation is a version of the discrete Painlevé equation dP_{II}. In fact, from this difference equation one can derive the Painlevé equation P_{II} by taking an appropriate continuum limit.

The parallel translations T_1, T_2, T_3 by the three vectors $\overrightarrow{P_0P_1}$. $\overrightarrow{P_1P_2}$. $\overrightarrow{P_2P_0}$ are expressed as

$$T_1 = \pi s_2 s_1, \quad T_2 = s_1 \pi s_2, \quad T_3 = s_1 s_2 \pi.$$

Furthermore, they satisfy the relations $T_iT_j = T_jT_i$ $(i, j = 1, 2, 3)$ and $T_1T_2T_3 = 1$. Note that the extended affine Weyl group $\widetilde{W} = \langle s_0, s_1, s_2, \pi \rangle$ is expressed as a semidirect product of a free abelian group $\mathcal{T}$ of rank 2 and the symmetric group $\mathfrak{S}_3 = \langle s_1, s_2 \rangle$ acting on $\mathcal{T}$:

$$\widetilde{W} = \langle s_0, s_1, s_2, \pi \rangle = \mathcal{T} \rtimes \mathfrak{S}_3, \qquad \mathcal{T} = \{ \, T_1^{k_1} T_2^{k_2} \mid k_1, k_2 \in \mathbb{Z} \, \}.$$

It means that, through the representation on $\mathcal{K} = \mathbb{C}(\alpha_0, \alpha_1, \alpha_2; f_0, f_1, f_2)$, the group of parallel translations of $\widetilde{W} = \langle s_0, s_1, s_2, \pi \rangle$ gives rise to a set of commuting dP_{II} flows.

Reference:

M. Noumi: *Painlevé equations through symmetry*,
Translations of Mathematical Monographs **223**, American Mathematical Society, 2004.

Discrete Painlevé equations

Consider a system of non-autonomous rational discrete evolution equations for two unknown functions (dependent variables) $f = f(x)$ and $g = g(x)$:

$$f(x+\delta) = R(x; f(x), g(x)), \qquad g(x+\delta) = S(x; f(x), g(x)),$$

where $R(x; f, g), S(x; f, g) \in \mathcal{K}(f, g)$ are rational functions in the variables (f, g) with coefficients in a difference field of "known functions". Typically, we consider the following three classes of discrete Painlevé equations, according to the choice of $\mathcal{K}$:

(dP):	$\mathcal{K} = \mathbb{C}(x)$	$\cdots$	rational/additive
(qP):	$\mathcal{K} = \mathbb{C}(e^x)$	$\cdots$	trigonometric/multiplicative ($q = e^\delta$)
(eP):	$\mathcal{K} = \mathbb{C}(\wp(x), \wp'(x))$	$\cdots$	elliptic

How to choose $R(x; f, g)$ and $S(x; f, g)$? How to detect discrete Painlevé equations?

- **Singularity confinement property:** Discrete counterpart of the Painlevé property

  Grammaticos, Ramani, Papageorgiou, Hietarinta, ... (Phys.Rev.Lett. 1991,...)
- **Algebraic entropy:** The degrees of the iterates of the rational map should grow polynomially.

  Bellon-Viallet (Com.Math.Phys. 1999),...
- **Birational representations of affine Weyl groups:**

  Noumi, Yamada, Kajiwara (Comm.Math.Phys. 1998, ...)
- **Geometry of rational surfaces:**

  Discrete dynamical systems on rational surfaces obtained from $\mathbb{P}^2$ by blowing up at nine points

  $\Longrightarrow$ Sakai's table of discrete Painlevé equations (Sakai (Comm.Math.Phys. 2001), ...)

Sakai's table of discrete Painlevé equations (of second order)

Sakai (2001) proposed a class of discrete Painlevé equations of second order, by classifying rational surfaces obtained from $\mathbb{P}^2$ by blowing-up at nine points (or from $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing-up at eight points) whose group of Cremona transformations form affine Weyl groups.

- In terms of the type of rational surfaces (anti-canonical divisors):

$$(eP): \quad A_0^{(1)}$$

$$\begin{array}{lllllllllllllllll} (qP): & A_0^{(1)} & \to & A_1^{(1)} & \to & A_2^{(1)} & \to & A_3^{(1)} & \to & A_4^{(1)} & \to & A_5^{(1)} & \to & A_6^{(1)} & \to & A_7^{(1)} & \to A_8^{(1)} \\ & & & & & & & & & & & & & & \searrow & & \\ & & & & & & & & & & & & & & & {A'}_7^{(1)} & \end{array}$$

$$\begin{array}{llllllllllllll} (dP): & A_0^{(1)} & \to & A_1^{(1)} & \to & A_2^{(1)} & \to & D_4^{(1)} & \to & D_5^{(1)} & \to & D_6^{(1)} & \to & D_7^{(1)} \to D_8^{(1)} \\ & & & & & & & & & & \searrow & & \searrow & \\ & & & & & & & & & & & E_6^{(1)} & \to & E_7^{(1)} \to E_8^{(1)} \end{array}$$

- In terms of the affine Weyl group symmetry:

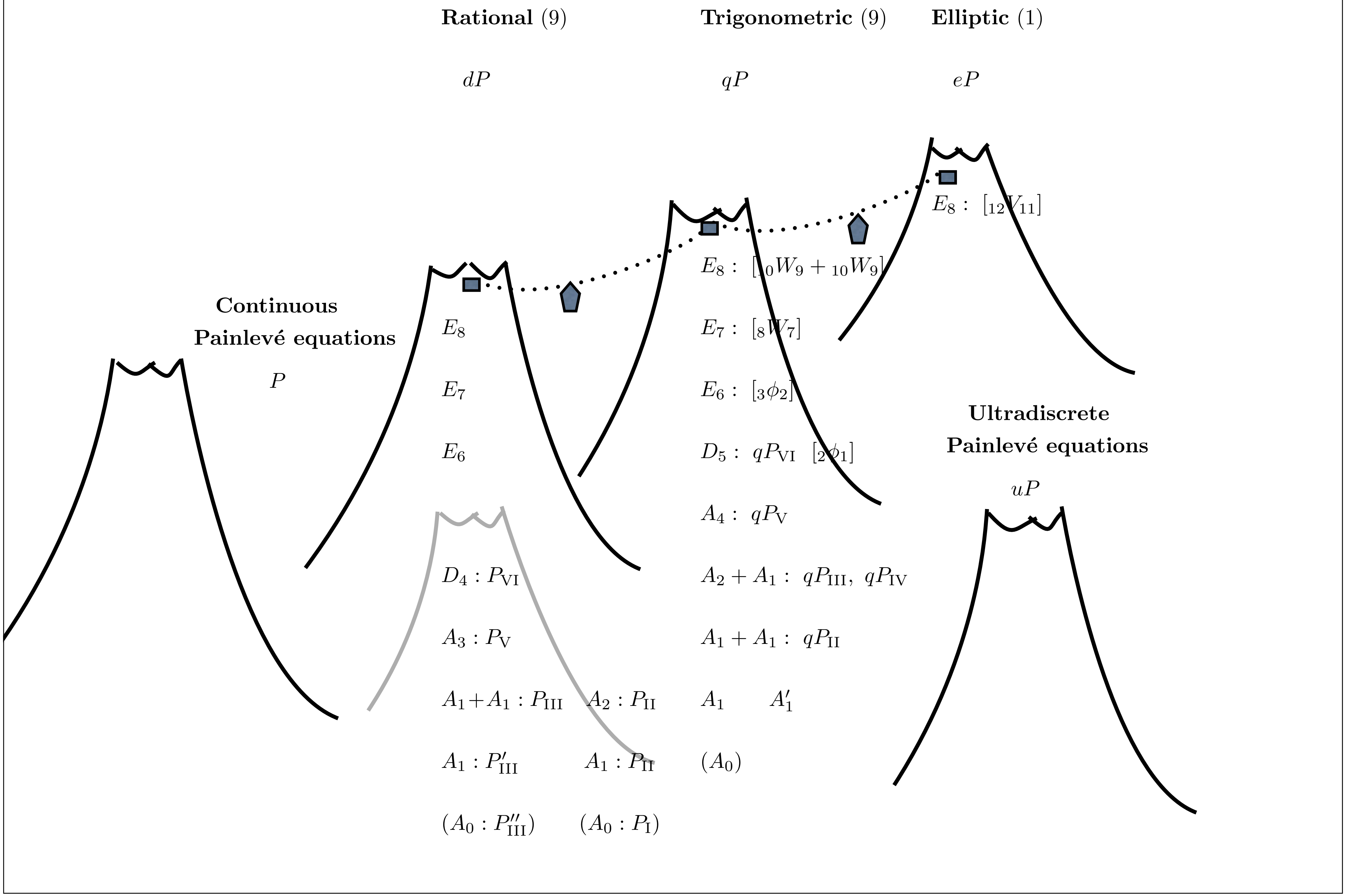
$$(eP): \quad E_8^{(1)}$$

$$\begin{array}{lllllllllllllllll} (qP): & E_8^{(1)} & \to & E_7^{(1)} & \to & E_6^{(1)} & \to & D_5^{(1)} & \to & A_4^{(1)} & \to & (A_2+A_1)^{(1)} & \to & (A_1+A_1')^{(1)} & \to & {A_1'}^{(1)} & \to A_0^{(1)} \\ & & & & & & & & & & & & & & \searrow & & \\ & & & & & & & & & & & & & & & A_1^{(1)} & \end{array}$$

$$\begin{array}{llllllllllllll} (dP): & E_8^{(1)} & \to & E_7^{(1)} & \to & E_6^{(1)} & \to & D_4^{(1)} & \to & A_3^{(1)} & \to & (2A_1)^{(1)} & \to & {A_1'}^{(1)} \to A_0^{(1)} \\ & & & & & & & & & & \searrow & & \searrow & \\ & & & & & & & & & & & A_2^{(1)} & \to & A_1^{(1)} \to A_0^{(1)} \end{array}$$

Discrete Painlevé equations

(Grammaticos-Ramani-$\cdots$ & Sakai)



An example: $qP(E_7^{(1)})$

q-Painlevé equation of type $E_7^{(1)}$ (Ramani-Grammaticos-Tamizhmani-Tamizhmani, Sakai, 2001)

- t: independent variable
- $b_1, b_2, \ldots, b_8$: parameters with $b_1b_2b_3b_4 = q$ and $b_5b_6b_7b_8 = 1$
- $f = f(t), g = g(t)$: unknown functions

Notation: $\overline{\varphi} = \varphi(qt)$, $\underline{\varphi} = \varphi(q^{-1}t)$ $\quad (\overline{b}_j = b_j, j = 1, \ldots, 8)$

$$\frac{(f\overline{g} - t\overline{t})(fg - t^2)}{(f\overline{g} - 1)(fg - 1)} = \frac{(f - b_1t)(f - b_2t)(f - b_3t)(f - b_4t)}{(f - b_5)(f - b_6)(f - b_7)(f - b_8)}$$

$$\frac{(fg - t^2)(\underline{f}g - \underline{t}t)}{(fg - 1)(\underline{f}g - 1)} = \frac{(g - t/b_1)(g - t/b_2)(g - t/b_3)(g - t/b_4)}{(g - 1/b_5)(g - 1/b_6)(g - 1/b_7)(g - 1/b_8)}$$

- When $b_1b_3 = b_5b_7$ (and $b_2b_4 = qb_6b_8$), this equation is decoupled consistently into the four equations (Murata-Sakai-Yoneda, 2003):

$$\frac{f\overline{g} - t\overline{t}}{f\overline{g} - 1} = \frac{(f - b_2t)(f - b_4t)}{(f - b_6)(f - b_8)}, \qquad \frac{fg - t^2}{fg - 1} = \frac{(f - b_1t)(f - b_3t)}{(f - b_5)(f - b_7)},$$

$$\frac{\underline{f}g - \underline{t}t}{\underline{f}g - 1} = \frac{(g - t/b_2)(g - t/b_4)}{(g - 1/b_6)(g - 1/b_8)}, \qquad \frac{fg - t^2}{fg - 1} = \frac{(g - t/b_1)(g - t/b_3)}{(g - 1/b_5)(g - 1/b_7)}.$$

Hence we obtain the *discrete Riccati equation* for g:

$$\overline{g} = \frac{(t\overline{t} - 1)f + t\left\{-(b_6 + b_8)\overline{t} + (b_2 + b_4)\right\}}{\{-(b_6 + b_8) + (b_2 + b_4)t\}\, f + b_6b_8(1 - t\overline{t})}, \quad f = \frac{(t^2 - 1)b_5b_7g + t\,\{(b_1 + b_3) - (b_5 + b_7)t\}}{\{t(b_1 + b_3) - (b_5 + b_7)\}\, g + (1 - t^2)}.$$

Namely, $\overline{g}$ is determined from g by a fractional linear transformation: $\overline{g} = \dfrac{Pg + Q}{Rg + S}$.

• Choose the variable

$$z=\frac{1-b_3/b_5t}{1-b_3/b_1}\frac{g-t/b_1}{g-1/b_5},\qquad \overline{z}=\frac{Az+B}{Cz+D}.$$

Then the corresponding discrete Riccati equation can be solved by

$$z=\frac{{}_8W_7(a;qb,c,d,e,f;q,x/q)}{{}_8W_7(a;b,c,d,e,f;q,x)},\quad x=\frac{q^2a^2}{bcdef};\quad (a;b,c,d,e,f)=\left(\frac{b_1b_8}{b_3b_5};\frac{b_8}{b_5},\frac{b_2}{b_3},\frac{b_4}{b_3},\frac{b_1t}{b_5},\frac{b_1}{b_5t}\right),$$

where ${}_{r+3}W_{r+2}$ denotes the very well-poised q-hypergeometric series defined by

$${}_{r+3}W_{r+2}(a_0;a_1,\ldots,a_r;q,x)=\sum_{k=0}^{\infty}\frac{1-q^{2k}a_0}{1-a_0}\frac{(a_0;q)_k(a_1;q)_k\cdots(a_r;q)_k}{(q;q)_k(qa_0/a_1;q)_k\cdots(qa_0/a_r;q)_k}x^k\quad(|q|<1,\quad |x|<1).$$

Here $(a;q)_k=(1-a)(1-qa)\cdots(1-q^{k-1}a)$ $(k=0,1,2,\ldots)$ stands for the standard q-shifted factorials.

• The discrete Riccati equation for z is linearized as

$$z=\frac{F}{G};\qquad \overline{F}=(AF+BG)H,\qquad \overline{G}=(CF+DG)H.$$

Hence we obtain two q-difference equations of second order:

$$\overline{F}+c_1F+c_2\underline{F}=0,\qquad \overline{G}+d_1G+d_2\underline{G}=0.$$

By choosing an appropriate gauge factor H, these equations for F and G are identified with the second order q-difference equations (three-term recurrence relations) for balanced ${}_8W_7$.

Reference:

K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada: Hypergeometric solutions to the q-Painlevé equations, IMRN **2004**:47(2004), 2497–2521.

Discrete Painlevé equations with $W(E_8^{(1)})$ symmetry

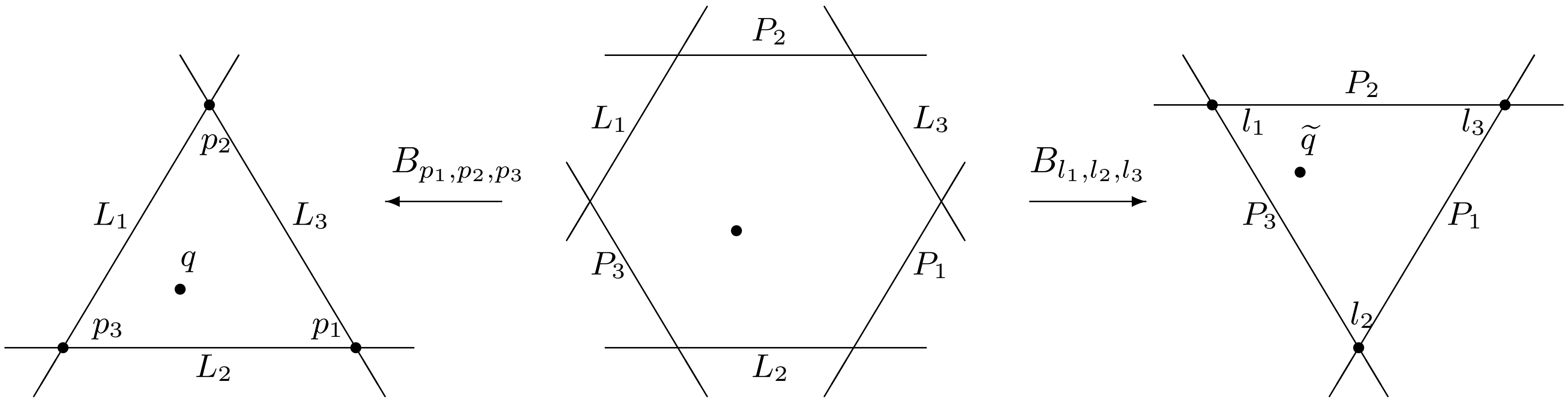
The three discrete Painlevé equations with $W(E_8^{(1)})$ symmetry can be formulated in terms of the configuration space of $(9+1)$ points in $\mathbb{P}^2$ (or $(8+1)$ points in $\mathbb{P}^1 \times \mathbb{P}^1$). They can be regarded as an non-autonomous version of *QRT mappings* (Quispel-Roberts-Thompson, Physica D. 1989).

Standard Cremona transformation (quadratic transformation)

Let p_1, p_2, p_3 be a triple of points in $\mathbb{P}^2$ which are not collinear. Then one can choose a system of homogeneous coordinates $(x_1 : x_2 : x_3)$ such that $p_1 = (1:0:0)$, $p_2 = (0:1:0)$, $p_3 = (0:0:1)$. Then the birational mapping

$$\mathrm{cr}_{p_1,p_2,p_3} : \quad \mathbb{P}^2 \cdots\rightarrow \mathbb{P}^2 : \quad q = (x_1 : x_2 : x_3) \rightarrow \widetilde{q} = (x_2x_3 : x_1x_3 : x_1x_2) = \left(\frac{1}{x_1} : \frac{1}{x_2} : \frac{1}{x_3}\right)$$

is called the *standard Cremona transformation* with respect to (p_1, p_2, p_3) (determined up to $(\mathbb{C}^*)^3$).



Configuration space $\mathbb{X}_{3,n}$ of n points in $\mathbb{P}^2$

We say that a n-tuple $(p_1, \ldots, p_n)$ of points in $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$ is *in general position* if, for any distinct triple $j_1, j_2, j_3 \in \{1, 2, \ldots, n\}$, the three points $p_{j_1}, p_{j_2}, p_{j_3}$ are not collinear. Two n-tuples $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n)$ are regarded as a same configuration if there exists a $g \in PGL(3;\mathbb{C})$ such that $g.p_j = q_j$ $(j = 1, \ldots, n)$. We consider the following configuration space of n points of $\mathbb{P}^2$ in general position:

$$\mathbb{X}_{3,n} = \{\ (p_1, \ldots, p_n) \in (\mathbb{P}^2)^n :\ \text{in general position}\ \}/\!\sim.$$

We denote by $[p_1, \ldots, p_n]$ the equivalence class ($PGL(3;\mathbb{C})$-orbit) of $(p_1, \ldots, p_n)$.

Note that the symmetric group $\mathfrak{S}_n$ of degree n acts on $\mathbb{X}_{3,n}$ from the right through the permutation of points:

$$[p_1, \ldots, p_n].\sigma = [p_{\sigma(1)}, \ldots, p_{\sigma(n)}] \qquad (\sigma \in \mathfrak{S}_n)$$

For each triple of distinct indices (j_1, j_2, j_3), we define the action of the standard Cremona transformation $\mathrm{cr}_{j_1,j_2,j_3}$ by

$$[p_1, \ldots, p_n].\mathrm{cr}_{j_1,j_2,j_3} = [q_1, \ldots, q_n]; \quad q_j = \begin{cases} p_j & (j \in \{j_1, j_2, j_3\}) \\ \mathrm{cr}_{p_{j_1},p_{j_2},p_{j_3}}(p_j) & (j \in \{1, \ldots, n\} \backslash \{j_1, j_2, j_3\}) \end{cases} .$$

Then it turns out that the group of birational transformations $\langle \{\mathrm{cr}_{j_1,j_2,j_3}\}_{1 \le j_1 < j_2 < j_3 \le n} ; \mathfrak{S}_n \rangle$ on $\mathbb{X}_{3,n}$, generated by the standard Cremona transformations $\mathrm{cr}_{j_1,j_2,j_3}$ together with $\mathfrak{S}_n$, gives a realization of the Weyl group $W(T_{2,3,n-3})$ associated with the tree $T_{2,3,n-3}$.

A system of coordinates of $\mathbb{X}_{3,n}$

If we fix homogeneous coordinates $(x_1 : x_2 : x_3)$ of $\mathbb{P}^2$, the configuration space $\mathbb{X}_{3,n}$ can be identified with the double coset space

$$\begin{aligned}&\mathrm{Mat}^*(3,n;\mathbb{C}) = \{\, X \in \mathrm{Mat}(3,n;\mathbb{C}) \mid \det(X)_{j_1,j_2,j_3} \neq 0 \quad (1 \leq j_1 < j_2 < j_3 \leq n)\},\\ &\mathbb{X}_{3,n} = GL(3;\mathbb{C})\backslash \mathrm{Mat}^*(3,n;\mathbb{C})/\mathbb{T}^n,\end{aligned}$$

where $\mathbb{T}^n = (\mathbb{C}^*)^n$ denotes the diagonal subgroup of $GL(n;\mathbb{C})$. By using the action of $GL(3;\mathbb{C})$ and $\mathbb{T}^n$, any $X \in \mathrm{Mat}^*(3,n;\mathbb{C})$ can be transformed into a unique canonical form:

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \end{bmatrix} \to Y = \begin{bmatrix} 1 & 0 & 0 & y_{14} & \cdots & y_{1n} \\ 0 & 1 & 0 & y_{24} & \cdots & y_{2n} \\ 0 & 0 & 1 & y_{34} & \cdots & y_{3n} \end{bmatrix} \to U = \begin{bmatrix} 1 & 0 & 0 & 1 & u_{15} & \cdots & u_{1n} \\ 0 & 1 & 0 & 1 & u_{25} & \cdots & u_{2n} \\ 0 & 0 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix},$$

where $u_{1j} = \xi_{124}\xi_{23j}/\xi_{12j}\xi_{234}$, $u_{1j} = \xi_{124}\xi_{13j}/\xi_{12j}\xi_{134}$, $\xi_{ijk} = \det(X)_{ijk}$. In this way we obtain a transversal $\mathcal{U}_{3,n} \xrightarrow{\sim} \mathbb{X}_{3,n} = GL(3;\mathbb{C})\backslash \mathrm{Mat}^*(3,n;\mathbb{C})/\mathbb{T}^n$, where

$$\mathcal{U}_{3,n} = \left\{ U = \begin{bmatrix} 1 & 0 & 0 & 1 & u_{15} & \cdots & u_{1n} \\ 0 & 1 & 0 & 1 & u_{25} & \cdots & u_{2n} \\ 0 & 0 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \;\middle|\; \det(U)_{j_1,j_2,j_3} \neq 0 \ (1 \leq j_1 < j_2 < j_3 \leq n) \right\}$$

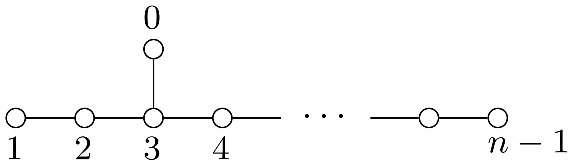
is a Zariski open subset of $\mathbb{C}^{2(n-4)}$. Hence, the field of rational functions on $\mathbb{X}_{3,n}$ is given by

$$\mathcal{K}(\mathbb{X}_{3,n}) = \mathbb{C}(u); \quad u = (u_{i,j})_{i=1,2;j=5,\ldots,n}.$$

Birational action of the Weyl group $W_{3,n}$ on $\mathbb{X}_{3,n}$

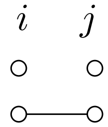
We denote by $W_{3,n} = W(T_{2,3,n-3}) = \langle s_0, s_1, \ldots, s_{n-1}\rangle$ the Weyl group associated with the Dynkin diagram $T_{2,3,n-3}$:

$T_{2,3,n-3}$:



Namely, $W_{3,n}$ is the group generated by the *simple reflections* $s_0, s_1, \ldots, s_{n-1}$ with the following fundamental relations:

$$W_{3,n} = \langle s_0, s_1, \ldots, s_{n-1}\rangle : \qquad s_i^2 = 1 \quad (i = 0, 1, \ldots, n-1); \qquad \begin{array}{l} s_i s_j = s_j s_i \\ s_i s_j s_i = s_j s_i s_j \end{array}$$



n	4	5	6	7	8	9	10	$\cdots$
root system	A_4	D_5	E_6	E_7	E_8	$E_8^{(1)}$	$*$	$\cdots$
$\dim_{\mathbb{C}} \mathbb{X}_{3,n}$	0	2	4	6	8	10	12	$\cdots$

($*$: of indefinite type)

This group $W_{3,n} = \langle s_0, s_1, \ldots, s_{n-1}\rangle$ acts on $\mathbb{X}_{3,n}$ birationally through the standard Cremona transformation $s_0 = \mathrm{cr}_{123}$ with respect to the first three points, and adjacent transpositions $s_1 = (12)$, $s_2 = (23)$, $\ldots$, $s_{n-1} = (n-1, n)$.

This birational action induces a realization of the Weyl group $W_{3,n}$ as a group of automorphisms of the field of rational functions $\mathcal{K}(\mathbb{X}_{3,n})$, so that

$$(w.\varphi)([p_1, \ldots, p_n]) = \varphi([p_1, \ldots, p_n].w)$$

for each $w \in W_{3,n}$, and for generic configurations $[p_1, \ldots, p_n] \in \mathbb{X}_{3,n}$.

In terms of the coordinates $u = (u_{ij})_{i=1,2;j=5,\dots,n}$ of the transversal $\mathcal{U}_{3,n}$ introduced above,

$$U = \begin{bmatrix} 1 & 0 & 0 & 1 & u_{15} & \cdots & u_{1n} \\ 0 & 1 & 0 & 1 & u_{25} & \cdots & u_{2n} \\ 0 & 0 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \in \mathcal{U}_{3,n}$$

the action of the simple reflections s_k $(k = 0, 1, \dots, n-1)$ of $W_{3,n}$ on $\mathcal{K}(\mathbb{X}_{3,n}) = \mathbb{C}(u)$ is computed explicitly as follows:

$$\begin{aligned}
k = 0: &\qquad s_0(u_{ij}) = \frac{1}{u_{ij}} \\
k = 1: &\qquad s_1(u_{1j}) = u_{2,j}, \quad s_1(u_{2j}) = u_{1j} \\
k = 2: &\qquad s_2(u_{1j}) = \frac{u_{1j}}{u_{2j}}, \quad s_2(u_{2j}) = \frac{1}{u_{2j}} \\
k = 3: &\qquad s_3(u_{ij}) = 1 - u_{ij} \\
k = 4: &\qquad s_4(u_{i5}) = \frac{1}{u_{i5}}, \quad s_4(u_{ij}) = \frac{u_{ij}}{u_{i5}} \quad (j = 6, \dots, n) \\
k = 5, \dots, n-1: &\qquad s_k(u_{ij}) = u_{i,s_k(j)}
\end{aligned}$$

Theorem: *The automorphisms $s_0, s_1, \dots, s_{n-1}$ of $\mathcal{K}(\mathbb{X}_{m,n}) = \mathbb{C}(u)$ defined as above satisfy the fundamental relations for the simple reflections of the Weyl group $W_{3,n} = W(T_{2,3,n-3})$.*

For any $w \in W_{3,n}$, the action of w on the coordinates u_{ij} is expressed as

$$w(u_{ij}) = R^w_{ij}(u) \quad (i = 1, 2;\ j = 5, \dots, n).$$

Since $w_1 w_2(u_{ij}) = w_1(w_2(u))$, these rational functions $R^w_{ij}(u)$ satisfy the compatibility condition

$$R^{w_1 w_2}_{ij}(u) = R^{w_2}_{ij}(R^{w_1}(u)) \quad (w_1, w_2 \in W_{3,8}); \qquad R^{w_1}(u) = (R^{w_1}_{kl}(u))_{kl}.$$

Canonical solution to the birational $W_{3,n}$ action on $\mathbb{X}_{3,n}$

Note that, for any generic configuration of nine points in $\mathbb{P}^2$, there exists a unique cubic curve (elliptic curve) passing through them. For general n, one can *solve* the system of $W_{3,n}$ equations for $u = (u_{ij})_{i=1,2;j=5,\ldots,n}$, by parametrizing the n points in terms of elliptic functions (or their degenerate cases). In fact, one can construct a set of functions $u_{ij} = u_{ij}(\varepsilon)$, in certain variables $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$ on which $W_{3,n}$ acts linearly, so that

$$u_{ij}(w(\varepsilon)) = R^w_{ij}(u(\varepsilon)) \quad (i = 1, 2;\ j = 5, \ldots, n;\ w \in W_{3,n}),$$

where $w(\varepsilon) = (w(\varepsilon_0), w(\varepsilon_1), \ldots, w(\varepsilon_n))$. For this purpose we make use of the Picard lattice $L_{3,n}$, and a particular realization of the affine root system of type $T_{2,3,n-3}$.

- **Picard lattice $L_{3,n}$**

The *Picard lattice* $L_{3,n}$ is a free $\mathbb{Z}$-module of rank $n+1$ with a *Lorentzian* symmetric bilinear form:

$$L_{3,n} = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n; \quad (e_0|e_0) = -1, \quad (e_j|e_j) = 1 \quad (j = 1, \ldots, n), \quad (e_i|e_j) = 0 \ \ (i \neq j).$$

In the algebro-geometric terms, $L_{3,n}$ is the Picard group of the rational surface obtained from $\mathbb{P}^2$ by blowing up at n points $p_1, \ldots, p_n$; e_0 represents the class of lines in $\mathbb{P}^2$, and $e_1, \ldots, e_n$ the exceptional curves corresponding to $p_1, \ldots, p_n$. For $\Lambda, \Lambda' \in L_{3,n}$, $(\Lambda|\Lambda') = -\Lambda \cdot \Lambda'$ denotes the minus of the intersection number of divisors.

• Realization of the root system of type $T_{2,3,n-3}$

We denote by $\mathfrak{h}_{3,n} = \mathbb{C} \otimes_{\mathbb{Z}} L_{3,n}$ the complexification of $L_{3,n}$, and by $\mathfrak{h}_{3,n}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}_{3,n}, \mathbb{C})$ its dual space; both $\mathfrak{h}_{3,n}$ and $\mathfrak{h}_{3,n}^*$ has a symmetric bilinear form induced from that of $L_{3,n}$:

$$\mathfrak{h}_{3,n} = \mathbb{C}e_0 \oplus \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n, \qquad \mathfrak{h}_{3,n}^* = \mathbb{C}\varepsilon_0 \oplus \mathbb{C}\varepsilon_1 \oplus \cdots \oplus \mathbb{C}\varepsilon_n,$$

where $\varepsilon_j = (\,\cdot\,|e_j) \in \mathfrak{h}_{3,n}^*$ $(j = 0, 1, \ldots, n)$. We reagard ε_j $(j = 0, 1, \ldots, n)$ as linear functions on $\mathfrak{h}_{3,n}$; the symmetric bilinear form on $\mathfrak{h}_{3,n}^*$ is given by

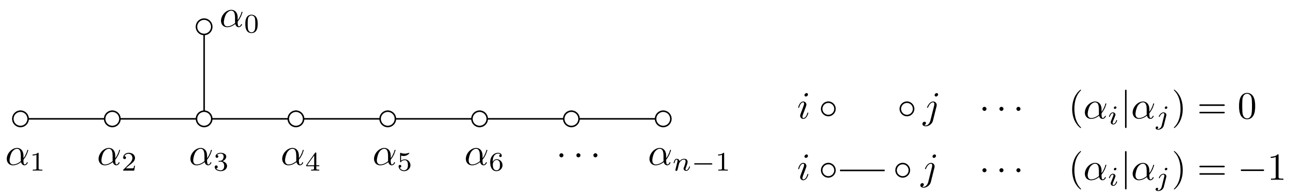
$$(\varepsilon_0|\varepsilon_0) = -1, \quad (\varepsilon_j|\varepsilon_j) = 1 \quad (j = 1, \ldots, n), \quad (\varepsilon_i|\varepsilon_j) = 0 \ \ (i \neq j).$$

In what follows, we use $\varepsilon = (\varepsilon_0; \varepsilon_1, \ldots, \varepsilon_n)$ as a system of coordinates for $\mathfrak{h}_{3,n}$.

We define the *simple roots* $\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in \mathfrak{h}_{3,n}^*$ by

$$\alpha_0 = \varepsilon_0 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3, \quad \alpha_j = \varepsilon_j - \varepsilon_{j+1} \quad (j = 1, \ldots, n-1).$$

Note that $(\alpha_j|\alpha_j) = 2$ for all $j = 0, 1, \ldots, n-1$, and that from the matrix $\big((\alpha_i, \alpha_j)\big)_{i,j=0}^{n-1}$, we obtain the Dynkin diagram of type $T_{2,3,n-3}$:



In the case $n = 9$ of affine root system of type $E_8^{(1)}$, in this realization the *null root* $\delta \in \mathfrak{h}_{3,9}^*$ is given by $\delta = 3\varepsilon_0 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7 - \varepsilon_8 - \varepsilon_9$. We also use the following notation of root lattices

$$Q(E_8^{(1)}) = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_7 \oplus \mathbb{Z}\alpha_8 \supset Q(E_8) = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_7.$$

• Action of the Weyl group $W_{3,n}$ on $\mathfrak{h}^*_{3,n}$

We define the *simple coroots* $h_j \in \mathfrak{h}_{3,n}$, by $h_0 = e_0 - e_1 - e_2 - e_3, \quad h_j = e_j - e_{j+1} \ \ (j = 1, \ldots, n-1)$, so that $\alpha_j = (\cdot|h_j)$. The Weyl group $W_{3,n}$ acts $\mathfrak{h}_{3,n}$ and $\mathfrak{h}^*_{3,n}$ in a standard manner:

$$s_j(h) = h - \langle h, \alpha_j\rangle h_j \ (h \in \mathfrak{h}_{3,n}); \quad s_j(\lambda) = \lambda - \langle h_j, \lambda\rangle\alpha_j \ (\lambda \in \mathfrak{h}^*_{3,n}).$$

We remark that the simple reflections s_0 acts on the linear functions $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n$ as follows.

$$\begin{aligned}
&s_0(\varepsilon_0) = 2\varepsilon_0 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3\\
&s_0(\varepsilon_1) = \varepsilon_0 - \varepsilon_2 - \varepsilon_3, \quad s_0(\varepsilon_2) = \varepsilon_0 - \varepsilon_1 - \varepsilon_3, \quad s_0(\varepsilon_3) = \varepsilon_0 - \varepsilon_1 - \varepsilon_2,\\
&s_0(\varepsilon_j) = \varepsilon_j \quad (j = 4, \ldots, n).
\end{aligned}$$

The other simple reflections s_k $(k = 1, \ldots, n-1)$ acts as $s_k(\varepsilon_0) = \varepsilon_0$ and $s_k(\varepsilon_j) = \varepsilon_{s_k(j)}$, $s_k = (k, k+1)$ as an adjacent transposition of indices.

In the case $n = 9$, we define the *canonical central element* $c \in \mathfrak{h}_{3,9}$ by $c = 3e_0 - e_1 - \cdots - e_9$, so that $\delta = (\cdot|c)$. This element $c \in \mathfrak{h}_{3,9}$ and the null root $\delta \in \mathfrak{h}^*_{3,9}$ are invariant under this action of $W(E_8^{(1)})$.

• Affine Weyl group and Kac's translations

We now consider the case $n = 9$ of affine root system of type $E_8^{(1)}$. For each element $\alpha \in \mathfrak{h}^*_{3,9}$ such that $(\delta|\alpha) = 0$, *Kac's translation* $T_\alpha : \mathfrak{h}^*_{3,9} \to \mathfrak{h}^*_{3,9}$ is defined by

$$T_\alpha(\lambda) = \lambda + (\delta|\lambda)\alpha - (\tfrac{1}{2}(\alpha|\alpha)(\delta|\lambda) + (\alpha|\lambda)) \quad (\lambda \in \mathfrak{h}^*_{3,9}).$$

When $\alpha, \beta \in \mathfrak{h}^*_{3,9}$ and $(\delta|\alpha) = (\delta|\beta) = 0$, one has $T_\alpha T_\beta = T_{\alpha+\beta}$; also, $wT_\alpha = T_{w.\alpha}w$ for any $w \in W_{3,9}$. When $\alpha \in Q(E_8^{(1)})$, T_α can be realized as an element of $W(E_8^{(1)})$. Furthermore one has

$$W(E_8^{(1)}) = T_Q \rtimes W(E_8); \quad Q(E_8) \xrightarrow{\sim} T_Q : \alpha \to T_\alpha.$$

- **Construction of a canonical solution**

Let $\Omega = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ be a free $\mathbb{Z}$-submodule of $\mathbb{C}$ generated by two complex numbers ω_1, ω_2 which are linearly independent over $\mathbb{R}$. We denote by $E_\Omega = \mathbb{C}/\Omega$ the corresponding elliptic curve, and by $\sigma(t)$ the Weierstrass sigma function associated with the period lattice Ω (possibly multiplied by e^{at^2+b}).

Regarding $\varepsilon = (\varepsilon_0; \varepsilon_1, \ldots, \varepsilon_n)$ as a system of coordinates for $\mathfrak{h}_{3,n}$, we fix a point $\varepsilon \in \mathfrak{h}_{3,n}$. If the coordinates $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{C}$ are generic, one can define a holomorphic mapping $p_\varepsilon : E_\Omega \to \mathbb{P}^2$ by

$$p_\epsilon(t) = \left(\frac{\sigma(\varepsilon_0 - \varepsilon_2 - \varepsilon_3 - t)}{\sigma(\varepsilon_1 - t)} : \frac{\sigma(\varepsilon_0 - \varepsilon_1 - \varepsilon_3 - t)}{\sigma(\varepsilon_2 - t)} : \frac{\sigma(\varepsilon_0 - \varepsilon_1 - \varepsilon_2 - t)}{\sigma(\varepsilon_3 - t)} \right) \quad (t \in \mathbb{C}),$$

so that the image $C_\varepsilon = p_\varepsilon(E_\Omega) \subset \mathbb{P}^2$ becomes a smooth cubic curve. By using this parametrization, we construct a meromorphic mapping

$$\varphi_{3,n} : \mathfrak{h}_{3,n} \cdots \to \mathbb{X}_{3,n} : \quad \varphi_{3,n}(\varepsilon) = [p_\varepsilon(\varepsilon_1), \cdots, p_\varepsilon(\varepsilon_n)] \in \mathbb{X}_{3,n}$$

by transferring the configuration of n points $\varepsilon_1, \ldots, \varepsilon_n$ on E_Ω to the cubic curve $C_\varepsilon \subset \mathbb{P}^2$.

Then it turns out that this meromorphic mapping $\varphi_{3,n}$ is $W_{3,n}$-equivariant with respect to the canonical linear action of $W_{3,n}$ on $\mathfrak{h}_{3,9}$ and the birational action on $\mathbb{X}_{3,9}$. In terms of the coordinates $u = (u_{ij})_{i=1,2;\ j=5,\ldots,n}$ of the transversal $\mathcal{U}_{3,n}$, $\varphi_{3,n}$ is expressed as

$$\varphi_{3,n} : \quad u_{ij} = u_{ij}(\varepsilon) = \frac{\sigma(\alpha_0 + \varepsilon_{3,4})\sigma(\varepsilon_{i4})}{\sigma(\alpha_0 + \varepsilon_{i4})\sigma(\varepsilon_{34})} \frac{\sigma(\alpha_0 + \varepsilon_{ij})\sigma(\varepsilon_{3j})}{\sigma(\alpha_0 + \varepsilon_{3j})\sigma(\varepsilon_{ij})} \quad (i = 1, 2;\ j = 5, \ldots, n)$$

where $\alpha_0 = \varepsilon_0 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3$, and $\varepsilon_{ij} = \varepsilon_i - \varepsilon_j$. These meromorphic function $u_{ij}(\varepsilon)$ *solve* the system of $W_{3,n}$ equations on $\mathbb{X}_{3,n}$:

$$u_{ij}(w(\varepsilon)) = R^w_{ij}(u(\varepsilon)) \qquad (i = 1, 2;\ j = 5, \ldots, n;\ w \in W_{3,n}).$$

Elliptic difference Painlevé equation: $eP(E_8^{(1)})$

In order to formulate discrete Painlevé equations with $W(E_8^{(1)})$ symmetry, we use the framework of the configuration space $\mathbb{X}_{3,10}$. In the configuration of ten points $[p_1, p_2, \ldots, p_9, q] \in \mathbb{X}_{3,10}$ $(q = p_{10})$, we use the first 9 points $p_1, \ldots, p_9$ as reference points for the standard Cremona transformations, and regard $q = p_{10}$ as the *general point* in $\mathbb{P}^2$ which is transformed by the action of $W_{3,8} = W(E_8^{(1)})$.

$$
\begin{array}{cccc}
[p_1, p_2, \ldots, p_9, q] & \in \mathbb{X}_{3,10} & \curvearrowleft\ W_{3,10} & (q = p_{10}) \\
 & \pi \downarrow & & \\
[p_1, p_2, \ldots, p_9] & \in \mathbb{X}_{3,9} & \curvearrowleft\ W_{3,9} &
\end{array}
$$

In the transversal $\mathcal{U}_{3,10}$, we distinguish the 10th column from the others and regard $(z_1, z_2) = (u_{1,10}, u_{2,10})$ as the inhomogeneous coordinates of a general point in $\mathbb{P}^2$:

$$
U_{3,10} = \begin{bmatrix} 1 & 0 & 0 & 1 & u_{15} & u_{16} & u_{17} & u_{18} & u_{19} & z_1 \\ 0 & 1 & 0 & 1 & u_{25} & u_{26} & u_{27} & u_{28} & u_{29} & z_2 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \in \mathcal{U}_{3,10} \xrightarrow{\sim} \mathbb{X}_{3,10},
$$

Then the action of $W_{3,9} \subset W_{3,10}$ on $\mathbb{X}_{3,10}$ is described as follows: for each $w \in W_{3,9}$,

$$
w(u_{ij}) = R^w_{ij}(u) \quad (i = 1, 2;\ j = 5, \ldots, 9); \qquad w(z_i) = S^w_i(u; z_1, z_2) \quad (i = 1, 2).
$$

In this system of birational transformations, we regard (z_1, z_2) as unknown functions, and $u = (u_{ij})_{i=1,2;\ j=5,\ldots,8}$ as parameters for the Cremona transformations. We have already seen that the system of $W_{3,9}$ equation for u_{ij} has a canonical solution $\varphi_{3,9} : \mathfrak{h}_{3,9} \cdots \to \mathbb{X}_{3,9}$.

Hence, by substituting this solution $u_{ij}(\varepsilon)$, we obtain a system of Cremona transformations

$$w(z_1) = S_1^w(\varepsilon; z_1, z_2), \quad w(z_2) = S_2^w(\varepsilon; z_1, z_2) \qquad (w \in W_{3,9})$$

for unknown functions z_1, z_2; here we have used the notation $S_i^w(\epsilon; z_1, z_2)$ in place of $S_i^w(u(\epsilon); z_1, z_2)$. This system can be described as a realization of $W_{3,9}$ as a group of automorphisms of the field of rational functions $\mathcal{K} = \mathbb{K}(z_1, z_2)$ with coefficients in the field $\mathbb{K} = \mathcal{M}(E_\Omega \otimes_{\mathbb{Z}} L_{3,9})$ of $\Omega \otimes_{\mathbb{Z}} L_{3,9}$-periodic meromorphic functions on $\mathfrak{h}_{3,9}$.

On this field $\mathcal{K} = \mathbb{K}(z_1, z_2)$, the action of the simple reflections $s_0, s_1, \ldots, s_8$ are determined as follows:

	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
z_1	$\dfrac{1}{z_1}$	z_2	$\dfrac{z_1}{z_2}$	$1 - z_1$	$\dfrac{z_1}{u_{15}(\varepsilon)}$	z_1	z_1	z_1	z_1
z_2	$\dfrac{1}{z_2}$	z_1	$\dfrac{1}{z_2}$	$1 - z_2$	$\dfrac{z_2}{u_{25}(\varepsilon)}$	z_2	z_2	z_2	z_2

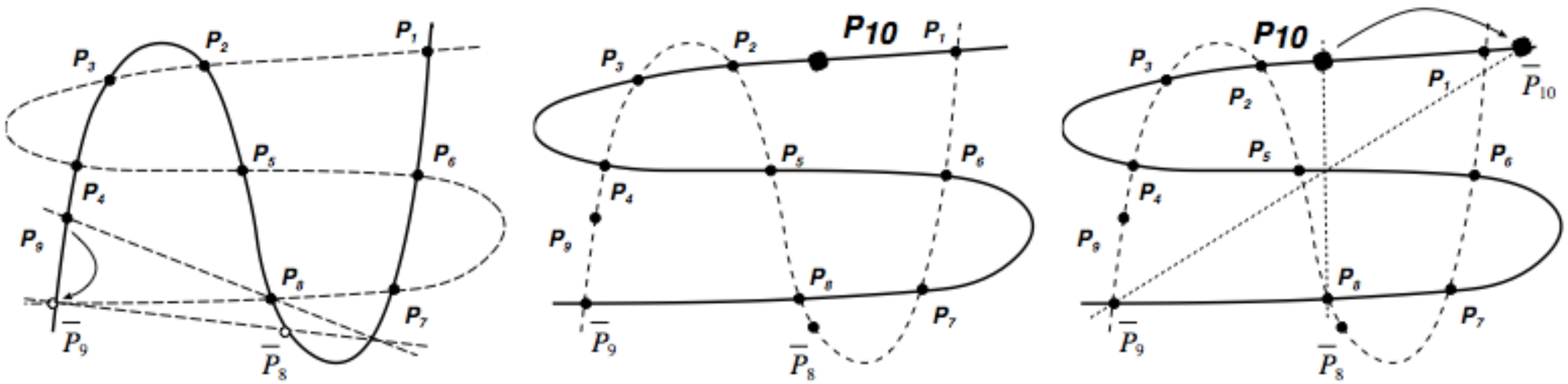
$$u_{15}(\varepsilon) = \frac{\sigma(\varepsilon_{14})\sigma(\varepsilon_{35})\sigma(\varepsilon_{124})\sigma(\varepsilon_{235})}{\sigma(\varepsilon_{34})\sigma(\varepsilon_{15})\sigma(\varepsilon_{234})\sigma(\varepsilon_{125})}, \quad u_{25}(\varepsilon) = \frac{\sigma(\varepsilon_{24})\sigma(\varepsilon_{35})\sigma(\varepsilon_{124})\sigma(\varepsilon_{135})}{\sigma(\varepsilon_{34})\sigma(\varepsilon_{25})\sigma(\varepsilon_{234})\sigma(\varepsilon_{125})}. \quad \begin{array}{l} \varepsilon_{ij} = \varepsilon_i - \varepsilon_j \\ \varepsilon_{ijk} = \varepsilon_0 - \varepsilon_i - \varepsilon_j - \varepsilon_k \end{array}.$$

Since $W_{3,9} = W(E_8^{(1)}) = T_Q \rtimes W(E_8)$, $T_Q = \{T_\alpha |\ \alpha \in Q(E_8)\}$, form the translation part we obtain a commuting family of birational transformations

$$T_\alpha(z_1) = S_1^\alpha(\varepsilon; z_1, z_2), \quad T_\alpha(z_2) = S_2^\alpha(\varepsilon; z_1, z_2) \qquad (\alpha \in Q(E_8))$$

parameterized by the root lattice $Q(E_8)$ of type E_8. This system of discrete time evolutions is the *elliptic difference Painelvé equation* $eP(E_8^{(1)})$. Note that $qP(E_8^{(1)})$ (resp. $dP(E_8^{(1)})$) can be obtained simply by replacing $\sigma(t)$ by $\sin(t)$ (resp. by t).

Geometric description of T_{α_8}, $\alpha_8 = \varepsilon_8 - \varepsilon_9$



Lattice τ-functions for $eP(E_8^{(1)})$

We formulated the elliptic difference Painlevé equation $eP(E_8^{(1)})$ as an action of $W_{3,9}$ on the field of rational functions $\mathcal{K} = \mathbb{K}(z_1, z_2)$ as a group of automorphisms. Here $\mathbb{K}$ denotes the field of $\Omega \otimes_{\mathbb{Z}} L_{3,9}$-periodic meromorphic functions on $\mathfrak{h}_{3,9}$. In order to understand the structure of $eP(E_8^{(1)})$, we introduce a system of homogeneous coordinates $(f_1 : f_2 : f_3)$ for $\mathbb{P}^2$ such that

$$z_1 = \frac{\sigma(\varepsilon_{12})\sigma(\varepsilon_{124})}{\sigma(\varepsilon_{34})\sigma(\varepsilon_{234})}\frac{f_1}{f_3}, \qquad z_2 = \frac{\sigma(\varepsilon_{24})\sigma(\varepsilon_{124})}{\sigma(\varepsilon_{34})\sigma(\varepsilon_{134})}\frac{f_2}{f_3},$$

together with new dependent variables $\tau_1, \ldots, \tau_9$ corresponding to the nine points $p_1, \ldots, p_9$. Then the action of $W_{3,9}$ on $\mathcal{K} = \mathbb{K}(z_1, z_2)$ can be extended to the field $\mathcal{L} = \mathbb{K}(f_1, f_2, f_3; \tau_1, \ldots, \tau_9)$ as follows:

$$\begin{aligned} s_0(\tau_i) &= f_i\,\tau_i && (i = 1,2,3), \\ s_0(\tau_j) &= \tau_j && (j = 4,\ldots,9), \\ s_k(\tau_j) &= \tau_{(k,k+1)j} && (k = 1,\ldots,8; j = 1,\ldots,9) \end{aligned}$$

$$\begin{aligned} s_0(f_i) &= \frac{1}{f_i} && (i = 1,2,3), \\ s_k(f_i) &= f_{(k,k+1)i} && (k = 1,2; i = 1,2,3), \\ s_k(f_i) &= f_i && (k = 4,\ldots,8; i = 1,2,3). \end{aligned}$$

$$\begin{aligned} s_3(f_1) &= \frac{\tau_3}{\tau_4}\left(\frac{\sigma(\varepsilon_{14})\sigma(\varepsilon_{124})}{\sigma(\varepsilon_{123})\sigma(\varepsilon_{13})}f_1 - \frac{\sigma(\varepsilon_{34})\sigma(\varepsilon_{234})}{\sigma(\varepsilon_{123})\sigma(\varepsilon_{13})}f_3\right), \\ s_3(f_2) &= \frac{\tau_3}{\tau_4}\left(\frac{\sigma(\varepsilon_{24})\sigma(\varepsilon_{124})}{\sigma(\varepsilon_{123})\sigma(\varepsilon_{23})}f_2 - \frac{\sigma(\varepsilon_{34})\sigma(\varepsilon_{134})}{\sigma(\varepsilon_{123})\sigma(\varepsilon_{23})}f_3\right), \\ s_3(f_3) &= \frac{\tau_3}{\tau_4}f_3. \end{aligned}$$

Theorem A: *The automorphisms $s_0, s_1, \ldots, s_8$ of $\mathcal{L} = \mathbb{K}(f_1, f_2, f_3; \tau_1, \ldots, \tau_9)$ defined as above satisfy the fundamental relations for the simple reflections of $W_{3,9} = \langle s_0, s_1, \ldots, s_8\rangle$.*

In this realization we look at the action of s_3 on f_1:

$$s_3(f_1) = \frac{\tau_3}{\tau_4}\left(\frac{\sigma(\varepsilon_{14})\sigma(\varepsilon_{124})}{\sigma(\varepsilon_{123})\sigma(\varepsilon_{13})}f_1 - \frac{\sigma(\varepsilon_{34})\sigma(\varepsilon_{234})}{\sigma(\varepsilon_{123})\sigma(\varepsilon_{13})}f_3\right).$$

By using the relations $f_i = s_0(\tau_i)/\tau_i$ $(i = 1, 2, 3)$, this formula can be rewritten as

$$\frac{s_3 s_0(\tau_1)}{\tau_1} = \frac{\tau_3}{\tau_4}\left(\frac{\sigma(\varepsilon_{14})\sigma(\varepsilon_{124})}{\sigma(\varepsilon_{123})\sigma(\varepsilon_{13})}\frac{s_0(\tau_1)}{\tau_1} - \frac{\sigma(\varepsilon_{34})\sigma(\varepsilon_{234})}{\sigma(\varepsilon_{123})\sigma(\varepsilon_{13})}\frac{s_0(\tau_3)}{\tau_3}\right).$$

This implies the following bilinear relations for translates of τ functions:

$$\sigma(\varepsilon_{123})\sigma(\varepsilon_{13})\tau_4\, s_3 s_0(\tau_1) = \sigma(\varepsilon_{14})\sigma(\varepsilon_{124})\tau_3 s_0(\tau_1) - \sigma(\varepsilon_{34})\sigma(\varepsilon_{234})\tau_1 s_0(\tau_3).$$

In order to analyze the action of $W_{3,9}$ on τ functions, we consider the $W_{3,9}$-orbit of e_9 in the Picard lattice $L_{3,9}$: $M_{3,9} = W_{3,9}\, e_9 \subset L_{3,9}$. This orbit can also be described intrinsically as

$$M_{3,9} = \{\ \Lambda \in L_{3,9} \mid (\Lambda|\Lambda) = 1,\ (\Lambda|c) = -1\ \}; \quad Q(E_8) \xrightarrow{\sim} M_{3,9} : \quad \alpha \mapsto T_\alpha(e_9).$$

Theorem B: *There exists a unique family of elements $\tau(\Lambda) \in \mathcal{L}$ $(\Lambda \in M_{3,9})$ such that*

$$\tau(e_j) = e_j \quad (j = 1, \ldots, 9); \quad w(\tau(\Lambda)) = \tau(w.\Lambda) \quad (\Lambda \in M_{3,9};\ w \in W_{3,9}).$$

Furthermore, this family of τ-functions is characterized by the following non-autonomous Hirota-Miwa equations: For any distinct $i, j, k, l \in \{1, \ldots, 9\}$,

$$\begin{aligned}\sigma(\varepsilon_{jkl})\sigma(\varepsilon_{jk})\tau(e_i)\tau(e_0 - e_l - e_i) + \sigma(\varepsilon_{kil})\sigma(\varepsilon_{ki})\tau(e_j)\tau(e_0 - e_l - e_j)\\ + \sigma(\varepsilon_{ijl})\sigma(\varepsilon_{ij})\tau(e_k)\tau(e_0 - e_l - e_k) = 0.\end{aligned}$$

For each $\Lambda \in M_{3,9}$ we define $\tau(\Lambda) = w(\tau_9) \in \mathcal{L}$ by taking a $w \in W_{3,9}$ such that $\Lambda = w.e_9$; this definition does not depend on the choice of w since τ_9 is invariant under the action of the isotropy subgroup $W_{3.8}$ of e_9. With this definition, the bilinear relation

$$\sigma(\varepsilon_{123})\sigma(\varepsilon_{13})\tau_4\, s_3 s_0(\tau_1) = \sigma(\varepsilon_{14})\sigma(\varepsilon_{124})\tau_3 s_0(\tau_1) - \sigma(\varepsilon_{34})\sigma(\varepsilon_{234})\tau_1 s_0(\tau_3).$$

is rewritten in the form

$$\begin{aligned}&\sigma(\varepsilon_{123})\sigma(\varepsilon_{13})\tau(e_4)\,\tau(e_0 - e_2 - e_4)\\ &= \sigma(\varepsilon_{14})\sigma(\varepsilon_{124})\tau(e_3)\tau(e_0 - e_2 - e_3) - \sigma(\varepsilon_{34})\sigma(\varepsilon_{234})\tau(e_1)\tau(e_0 - e_1 - e_2).\end{aligned}$$

Then by the action of $\mathfrak{S}_9$ we obtain the bilinear equations as described in Theorem B.
Conversely, suppose that the family $\tau(\Lambda)$ $(\Lambda \in M_{3,9})$ satisfies the property as stated in Theorem B. Then the variables f_i $(i = 1, 2, 3)$ are recovered by

$$f_1 = \frac{\tau(e_0 - e_2 - e_3)}{\tau(e_1)}, \quad f_2 = \frac{\tau(e_0 - e_1 - e_3)}{\tau(e_2)}, \quad f_3 = \frac{\tau(e_0 - e_1 - e_2)}{\tau(e_3)}.$$

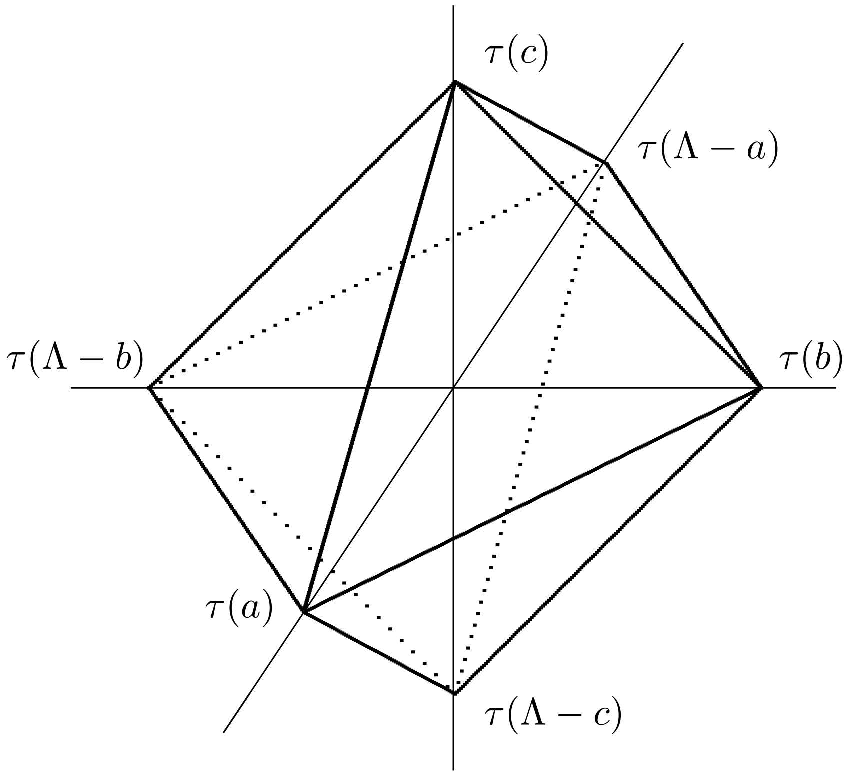
The non-autonomous Hirota-Miwa equations mentioned above guarantee the validity of relations to be satisfied under the action of s_3.

Non-autonomous Hirota-Miwa equation

attached to a C_3-frame (octahedron)

$$\sigma(\beta-\gamma)\sigma(\lambda-\beta-\gamma)\ \tau(a)\tau(\Lambda-a)+\ \sigma(\gamma-\alpha)\sigma(\lambda-\gamma-\alpha)\ \tau(b)\tau(\Lambda-b)$$
$$+\ \sigma(\alpha-\beta)\sigma(\lambda-\alpha-\beta)\ \tau(c)\tau(\Lambda-c)=0$$

$$\lambda=(\cdot|\Lambda),\quad \alpha=(\cdot|a),\quad \beta=(\cdot|b),\quad \gamma=(\cdot|c).$$



$$a,b,c,\Lambda-a,\Lambda-b,\Lambda-c\in M_{3,9}$$

Hypergeometric τ-function of $eP(E_8^{(1)})$

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In the following, we present a special *hypergeometric* solution of $eP(E_8^{(1)})$. We assume that the period lattice Ω is given by $\Omega = \mathbb{Z}1 \oplus \mathbb{Z}\omega$ with $\omega \in \mathbb{C}$ with $\mathrm{Im}\,(\omega) > 0$, and set $p = e(\omega) = \exp(2\pi\sqrt{-1}\omega)$, so that $|p| < 1$. We use the following theta function for $\sigma(u)$:

$$\sigma(u) = z^{-\frac{1}{2}}\theta(z;p), \quad z = e(u); \quad \theta(z;p) = (z;p)_\infty (p/z;p)_\infty, \quad (z;p)_\infty = \prod_{i=0}^{\infty}(1 - p^i z).$$

Also, regarding the null root δ as a constant with $\mathrm{Im}\,(\delta) > 0$, we set $q = e(\delta)$, $|q| < 1$.

Elliptic gamma functions (Ruijsenaars)

Recall that the elliptic gamma functions $\Gamma(z;p,q)$ and $\Gamma(z;p,q,r)$ are defined by

$$\Gamma(z;p,q) = \frac{(pq/z;p,q)}{(z;p,q)_\infty}, \qquad \Gamma(z;p,q,r) = (z;p,q,r)_\infty (pqr/z;p,q,r)_\infty$$

under the conditions $|p| < 1, |q| < 1, |r| < 1$, where

$$(z;p,q)_\infty = \prod_{i,j=0}^{\infty}(1 - p^i q^j z), \quad (z;p,q,r)_\infty = \prod_{i,j,k=0}^{\infty}(1 - p^i q^j r^k z).$$

These functions satisfy the functional equations

$$\begin{aligned} \theta(pz;p) &= -z^{-1}\theta(z;p), & \theta(p/z;p) &= \theta(z;p), \\ \Gamma(qz;p,q) &= \theta(z;p)\Gamma(z;p,q), & \Gamma(pq/z;p,q) &= \Gamma(z;p,q)^{-1}, \\ \Gamma(rz;p,q,r) &= \Gamma(z;p,q)\Gamma(z;p,q,r), & \Gamma(pqr/z;p,q) &= \Gamma(z;p,q,r). \end{aligned}$$

Elliptic hypergeometric integrals

Following Spiridonov and Rains, for $t = (t_0, t_1, \dots, t_7) \in \mathbb{T}^8$, we consider the *elliptic hypergeometric integrals*

$$I(t;p,q) = \frac{(p;p)_\infty (q;q)_\infty}{2 \cdot 2\pi\sqrt{-1}} \int_{C(t)} \frac{\prod_{i=0}^{7} \Gamma(t_i z_k^{\pm 1};p,q)}{\Gamma(z_k^{\pm 2};p,q)} \frac{dz}{z},$$

where $\Gamma(tz^{\pm 1};p,q) = \Gamma(tz;p,q)\Gamma(tz^{-1};p,q)$. Also, for $n = 0, 1, 2, \dots$, we set

$$I^{(n)}(t;p,q) = \frac{(p;p)_\infty^n (q;q)_\infty^n}{2^n n! (2\pi\sqrt{-1})^n} \int_{C_n(t)} \prod_{k=1}^{n} \frac{\prod_{i=0}^{7} \Gamma(t_i z_k^{\pm 1};p,q)}{\Gamma(z_k^{\pm 2};p,q)} \prod_{1 \le k < l \le n} \theta(z_k^{\pm 1} z_l^{\pm 1};p) \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n}.$$

- **Elliptic Bailey transformation (Spiridonov): $\boldsymbol{W(E_7)}$ symmetry**

When $t_0 t_1 \cdots t_7 = p^2 q^2$, one has

$$I(t;p,q) = I(\widetilde{t};p,q) \prod_{0 \le i < j \le 3} \Gamma(t_i t_j;p,q) \prod_{4 \le i < j \le 7} \Gamma(t_i t_j;p,q); \quad \widetilde{t}_i = \begin{cases} t_i \sqrt{pq/t_0 t_1 t_2 t_3} & (i = 0,1,2,3), \\ t_i \sqrt{pq/t_4 t_5 t_6 t_7} & (i = 4,5,6,7). \end{cases}$$

- **Three-term recurrence relations**

Introducing additive variables $x = (x_0, x_1, \dots, x_7)$, $t_i = e(x_i)$ $(i = 0, 1, \dots, 7)$, we modify $I(t;p,q)$ as

$$J(x) = e(-(x|x)/2\delta)\, I(t;p,q); \quad (x|x) = x_0^2 + x_1^2 + \cdots + x_7^2.$$

Then $J(x)$ satisfies the *three-term recurrence relations*

$$\sigma(x_j \pm x_k)\, T_{x_i}^{\delta} J(x) + \sigma(x_k \pm x_i)\, T_{x_j}^{\delta} J(x) + \sigma(x_i \pm x_j)\, T_{x_k}^{\delta} J(x) = 0$$

for all $i, j, k \in \{0, 1, \dots, 7\}$, where $\sigma(a \pm b) = \sigma(a+b)\sigma(a-b)$.

Hypergeometric τ-function

Denoting $\phi = \delta - \alpha_8$ the highest root of the root system E_8, we consider to construct a special solution of the elliptic difference Painlevé equation $eP(E_8^{(1)})$ under the restriction $\phi = \omega$, $\sigma(\phi) = 0$.

We relate the coordinates $\varepsilon = (\varepsilon_0; \varepsilon_1, \dots, \varepsilon_9)$ for $\mathfrak{h}_{3,9}$ to the additive variables $x = (x_0, x_1, \dots, x_7)$ by

$$x_i = \varepsilon_i - \tfrac{1}{2}(\varepsilon_0 - \varepsilon_9) + \tfrac{1}{2}\delta \quad (i = 1, \dots, 8); \quad x_0 = -x_8.$$

In these variables, the highest root is expressed as $\phi = \phi(x) = \frac{1}{2}(x_0 + x_1 + \cdots + x_7)$. We also define the multiplicative variables $t = (t_0, \dots, t_7)$ by $t_i = e(x_i)$ $(i = 0, 1, \dots, 7)$.

Denoting by $V = \mathbb{C}^8$ the 8-dimensional affine space with canonical coordinates $x = (x_0, x_1, \dots, x_7)$. For each $n \in \mathbb{Z}$, we define the hyperplanes H_n $(n \in \mathbb{Z})$ by

$$H_n = \{\, x \in V \mid \phi(x) = \omega + n\delta \} \subset V,$$

For each $n \in \mathbb{Z}$, we define a meromorphic function $\tau^{(n)} = \tau^{(n)}(x)$ on H_n as follows:

$$\tau^{(n)}(x) \equiv 0 \qquad (n < 0),$$

$$\tau^{(0)}(x) = \prod_{0 \le i < j \le 7} \Gamma(q t_i t_j; p, q, q) \quad (x \in H_0),$$

$$\tau^{(1)}(x) = \prod_{0 \le i < j \le 7} \Gamma(t_i t_j; p, q, q)\; J(x), \quad J(x) = e(-(x|x)/2\delta)\, I(t; p, q), \quad (x \in H_1).$$

For general $n = 0, 1, 2, \ldots$, we define

$$\tau^{(n)}(x) = \prod_{0 \le i < j \le 7} \Gamma(q^{1-n} t_i t_j; p, q, q)\, J^{(n)}(x - \tfrac{1}{2}(n-1)\delta)$$
$$J^{(n)}(x) = (-1)^{\binom{n}{2}}\, e(-n(x|x)/2\delta)\, I^{(n)}(t; p, q) \quad (x \in H_n),$$

where

$$I^{(n)}(t; p, q) = \frac{(p;p)_\infty^n (q;q)_\infty^n}{2^n n! (2\pi\sqrt{-1})^n} \int_{C_n(t)} \prod_{k=1}^{n} \frac{\prod_{i=0}^{7} \Gamma(t_i z_k^{\pm 1}; p, q)}{\Gamma(z_k^{\pm 2}; p, q)} \prod_{1 \le k < l \le n} \theta(z_k^{\pm 1} z_l^{\pm 1}; p) \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n}.$$

Taking the disjoint union of hyperplanes H_n $(n \in \mathbb{Z})$, we set

$$D = \bigsqcup_{n \in \mathbb{Z}} H_n = \{\, x \in V \ ;\ \phi(x) \in \omega + \mathbb{Z}\delta \} \subset V.$$

Then the sequence of functions $\tau^{(n)} = \tau^{(n)}(x)$ $(n \in \mathbb{Z})$ define a meromorphic function $\tau = \tau(x)$ on D such that $\tau|_{H_n} = \tau^{(n)}$ $(n \in \mathbb{Z})$.

Through the parametrization

$$x_0 = -\varepsilon_8 + \tfrac{1}{2}(\varepsilon_0 - \varepsilon_9) - \tfrac{1}{2}\delta, \quad x_i = \varepsilon_i - \tfrac{1}{2}(\varepsilon_0 - \varepsilon_9) + \tfrac{1}{2}\delta \quad (i = 1, \ldots, 7),$$

$\tau(x)$ is regarded as a meromorphic function on

$$\mathcal{D} = \{\ h \in \mathfrak{h}_{3,9} \mid (h|c) = \delta, \quad \langle h, \phi \rangle \in \omega + \mathbb{Z}\delta\ \} \subset \mathfrak{h}_{3,9}.$$

We also set $\mathcal{D}_0 = \{\ h \in \mathfrak{h}_{3,9} \mid (h|c) = \delta, \quad \langle h, \phi \rangle = \omega\ \} \subset \mathfrak{h}_{3,9}$.

Theorem: *With the meromorphic function* τ *on* $\mathcal{D}$ *defined as above, we set* $\tau(\Lambda) = T_{e_9 - \Lambda}(\tau)$ *for each* $\Lambda \in M_{3,9}$. *Then the meromorphic functions* $\tau(\Lambda)$ $(\Lambda \in M_{3,9})$ *on* $\mathcal{D}_0 \subset \mathfrak{h}_{3,9}$ *form a set of lattice* τ*-functions for* $eP(E_8^{(1)})$ *in the sense of Theorem B.*