

Semester 2, 2003

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Computer Tutorial 9

1. If G is a group and $a \in G$ then the *centralizer* of a is the set of all elements $g \in G$ that commute with a (meaning that $ag = ga$). Centralizers will also be discussed in Questions 4 and 5 of Tutorial 9, where we shall show that the centralizer of an element a is always a subgroup of G .

Let $G = \text{Sym}(11)$, and let $a \in G$ be the 9-cycle $(1, 2, 3, 4, 5, 6, 7, 8, 9)$.

- (i) Use MAGMA to construct the centralizer of a in G . (Define G first, then define the element a , and then use $C := \text{Centralizer}(G, a)$ to define the centralizer.)
- (ii) Get MAGMA to print C and the set of elements of C . (MAGMA distinguishes between the group and its set of elements, although mathematicians often do not. Use $\text{Set}(C)$ to obtain the set.)
- (iii) Define p to be $(2, 4, 5, 10, 11)(3, 6)$ (an element of G), and C_1 to be the coset Cp . Print out the elements of C_1 . (You can use $C_1 := \{ x * p : x \text{ in } C \}$.)
- (iv) Use a for loop to print out the values of $x^{-1}ax$ for all $x \in C_1$. (You can use `for x in C1 do x^(-1)*a*x; end for;`)
- (v) You should have noticed something. Can you explain it?
- (vi) Redefine p to be some other element of G —choose any one you like— and repeat the above three parts. Do it for at least 4 different values of p . Have you found an example for which $x^{-1}ax$ is not a 9-cycle? Can you?

Solution.

```
> G:=Sym(11);
> a:=G!(1,2,3,4,5,6,7,8,9);
> C:=Centralizer(G,a);
> C;
Permutation group C acting on a set of cardinality 11
Order = 18 = 2 * 3^2
  (10, 11)
  (1, 2, 3, 4, 5, 6, 7, 8, 9)
> Set(C);
{
  (1, 2, 3, 4, 5, 6, 7, 8, 9),
  (1, 4, 7)(2, 5, 8)(3, 6, 9)(10, 11),
```

```
(1, 8, 6, 4, 2, 9, 7, 5, 3)(10, 11),
(1, 2, 3, 4, 5, 6, 7, 8, 9)(10, 11),
(1, 4, 7)(2, 5, 8)(3, 6, 9),
(1, 8, 6, 4, 2, 9, 7, 5, 3),
(1, 6, 2, 7, 3, 8, 4, 9, 5),
(1, 3, 5, 7, 9, 2, 4, 6, 8)(10, 11),
(1, 3, 5, 7, 9, 2, 4, 6, 8),
(1, 6, 2, 7, 3, 8, 4, 9, 5)(10, 11),
(1, 5, 9, 4, 8, 3, 7, 2, 6)(10, 11),
(1, 5, 9, 4, 8, 3, 7, 2, 6),
Id(C),
(10, 11),
(1, 9, 8, 7, 6, 5, 4, 3, 2)(10, 11),
(1, 7, 4)(2, 8, 5)(3, 9, 6),
(1, 7, 4)(2, 8, 5)(3, 9, 6)(10, 11),
(1, 9, 8, 7, 6, 5, 4, 3, 2)
}
> p:=G!(2,4,5,10,11)(3,6);
> C1:={ c*p : c in C };
> C1;
{
  (1, 8, 3)(2, 9, 7, 10)(5, 6),
  (1, 6, 8)(2, 5, 7, 9, 4, 3, 10, 11),
  (2, 4, 5, 10, 11)(3, 6),
  (1, 10, 11, 2, 3, 7, 4, 8, 6)(5, 9),
  (1, 5, 8, 4, 7)(2, 10, 11)(6, 9),
  (1, 9, 8, 7, 3, 4, 6, 10, 11, 2),
  (1, 4, 10, 2, 6, 7, 8, 9)(3, 5),
  (1, 6, 8)(2, 5, 7, 9, 4, 3, 10),
  (1, 7, 5, 4)(2, 8, 10, 11)(3, 9),
  (1, 3, 8, 5)(2, 7, 6, 4, 9, 10),
  (2, 4, 5, 10)(3, 6),
  (1, 9, 8, 7, 3, 4, 6, 10, 2),
  (1, 10, 2, 3, 7, 4, 8, 6)(5, 9),
  (1, 4, 10, 11, 2, 6, 7, 8, 9)(3, 5),
  (1, 3, 8, 5)(2, 7, 6, 4, 9, 10, 11),
  (1, 5, 8, 4, 7)(2, 10)(6, 9),
  (1, 8, 3)(2, 9, 7, 10, 11)(5, 6),
  (1, 7, 5, 4)(2, 8, 10)(3, 9)
}
> for x in C1 do
for> x^(-1)*a*x;
for> end for;
(1, 4, 6, 5, 10, 3, 7, 8, 9)
(1, 4, 6, 5, 10, 3, 7, 8, 9)
```

```

(1, 4, 6, 5, 10, 3, 7, 8, 9)
(1, 4, 6, 5, 10, 3, 7, 8, 9)
(1, 4, 6, 5, 10, 3, 7, 8, 9)
(1, 4, 6, 5, 10, 3, 7, 8, 9)
(1, 4, 6, 5, 10, 3, 7, 8, 9)
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(1, 4, 6, 5, 10, 3, 7, 8, 9)
(1, 4, 6, 5, 10, 3, 7, 8, 9)
(1, 4, 6, 5, 10, 3, 7, 8, 9)
(1, 4, 6, 5, 10, 3, 7, 8, 9)
(1, 4, 6, 5, 10, 3, 7, 8, 9)
> p:=Random(G);
> p;
(1, 9, 5, 3)(2, 7, 10, 11, 6, 4)
> C1:={ c*p : c in C };
> for x in C1 do
for> x^(-1)*a*x;
for> end for;
(1, 2, 3, 4, 10, 8, 5, 9, 7)
(1, 2, 3, 4, 10, 8, 5, 9, 7)
... 15 lines omitted ...
(1, 2, 3, 4, 10, 8, 5, 9, 7)
> p:=Random(G);
> p;
(1, 2, 6, 5, 9, 7)(3, 10, 8, 4)
> C1:={ c*p : c in C };
> for x in C1 do
for> x^(-1)*a*x;
for> end for;
(1, 4, 7, 2, 6, 10, 3, 9, 5)
(1, 4, 7, 2, 6, 10, 3, 9, 5)
... 15 lines omitted ...
(1, 4, 7, 2, 6, 10, 3, 9, 5)
> p:=Random(G);
> p;
(3, 8, 9)(4, 11, 6, 10)(5, 7)
> C1:={ c*p : c in C };
> for x in C1 do
for> x^(-1)*a*x;

```

```

for> end for;
(1, 2, 8, 11, 7, 10, 5, 9, 3)
(1, 2, 8, 11, 7, 10, 5, 9, 3)
... 15 lines omitted ...
(1, 2, 8, 11, 7, 10, 5, 9, 3)

```

Observe that when asked to print C MAGMA returned brief but useful information about the group, notably its order and a set of generators. In this case the generators chosen by MAGMA consist of a 9-cycle and a 2-cycle, which act on disjoint sets: the two fixed points of the second generator are transposed by the first generator, and the nine fixed points of the first generator are cyclically permuted by the second. It follows from this that these generators commute with one another. If we denote the generators by u and t then the 18 elements of C are the 18 elements $u^i c^j$, where $0 \leq i < 2$ and $0 \leq j < 9$.

If p is a fixed element of G then the coset Cp consists of the 18 elements of the form cp , where $c \in C$. By the definition of C , each $c \in C$ satisfies $ac = ca$, or (equivalently) $c^{-1}ac = a$. If $x \in Cp$ then $x = cp$ for some $c \in C$, and so $x^{-1}ax = (cp)^{-1}a(cp) = p^{-1}c^{-1}acp = p^{-1}ap$. So all 18 elements $x \in Cp$ give the same value for $x^{-1}ax$.

If $x \in G$ then $x^{-1}(1, 2, 3, 4, 5, 6, 7, 8, 9)x = (1^x, 2^x, 3^x, 4^x, 5^x, 6^x, 7^x, 8^x, 9^x)$ (cf. solutions to Assignment 2). So $x^{-1}ax$ is always a 9-cycle. The total number of 9-cycles in G is $\binom{11}{9}8! = \frac{11!}{18}$, the same as $[G : C]$, the number of distinct right cosets of C in G . Each coset corresponds to a 9-cycle, which is the value of $x^{-1}ax$ for all x in the coset.

2. Set up the following group and find its order.

$G := \text{PermutationGroup} \langle 9 \mid (1, 2, 3)(7, 8, 9), (1, 2)(3, 4)(5, 6)(7, 8) \rangle$;
Define $x = (1, 2, 3)(7, 8, 9)$ and $y = (1, 2)(3, 4)(5, 6)(7, 8)$, the generators of G .

There is an equivalence relation on $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that numbers a and b are equivalent if and only if there is an element $g \in G$ such that $a^g = b$. The equivalence classes are called *orbits*. Two numbers are in the same orbit if you can get from one number to the other in a finite sequence of steps, where each step is to apply either x or y or their inverses. Thus $4^y = 3$ and $3^{x^{-1}} = 2$; so 4 and 2 are in the same orbit.

- (i) Determine whether the following pairs of numbers are in the same orbit of G : 1 and 2; 1 and 5; 2 and 4; 4 and 1; 5 and 8; 8 and 9.
- (ii) The notation a^G is often used for the orbit containing a . Thus, by definition, $a^G = \{a^g : g \in G\}$. To use MAGMA to find the orbit of 3 (for example), you can either type $\{3 \sim g : g \text{ in } G\}$ or $3 \sim G$. In either case you will see the orbit of 3. Try both commands and then use this method to check your answers to (i) above.
- (iii) For each number i from 1 to 9 find the order of the stabilizer of i in G . Recall that the MAGMA code to produce the stabilizer of 4 in G is

Stabilizer($G,4$).

- (iv) Can you see any relationship between the order of G , the order of the stabilizer of i and the number of elements in the same orbit as i ?
- (v) Let H be the stabilizer of 7 in G .
- What is the order of H ?
 - How many right cosets of H are there in G ?
 - Let g_1 be an element of G which takes 7 to 8 and let g_2 be an element of G which takes 7 to 9. Use MAGMA to construct the cosets $C_0 := H$, $C_1 := Hg_1$ and $C_2 := Hg_2$. For example,
 $> C2 := \{ h * g2 : h \text{ in } H \};$
 - Find out where each element of $C1$ takes 7. Do the same for $C2$.
 - You already have the answers to the following points in previous parts of this question.
 - What is the orbit of 7?
 - How many cosets does H have in G ?
 - What is the relationship (if any) between the cosets of H and the elements in the orbit of 7?

Solution.

$1^x = 2$; so 1 and 2 are in the same orbit. Since $5^y = 6$ and $6^y = 5$, and x fixes both 5 and 6, you cannot get from 5 to anywhere else but 5 and 6. The orbit containing 5 is $\{5,6\}$; 1 and 5 are not in the same orbit. Since $4^y = 3$ and $3^{x^{-1}} = 2$, 4 and 2 are in the same orbit. Since 1 and 2 are in the same orbit, and 2 and 4 are in the same orbit, there is no doubt that 1 and 4 are in the same orbit. We have already seen that the orbit containing 5 is $\{5,6\}$; so 5 and 8 are not in the same orbit. And $8^x = 9$; so 8 and 9 are in the same orbit.

```
> G:=PermutationGroup<9|(1,2,3)(7,8,9),(1,2)(3,4)(5,6)(7,8)>;
> #G;
72
> 3^G;
GSet{ 1, 2, 3, 4 }
> {3^g : g in G};
{ 1, 2, 3, 4 }
> for i in [1..9] do
for> "The orbit containing",i,"is",{i^g : g in G};
for> end for;
The orbit containing 1 is { 1, 2, 3, 4 }
The orbit containing 2 is { 1, 2, 3, 4 }
The orbit containing 3 is { 1, 2, 3, 4 }
```

```
The orbit containing 4 is { 1, 2, 3, 4 }
The orbit containing 5 is { 5, 6 }
The orbit containing 6 is { 5, 6 }
The orbit containing 7 is { 7, 8, 9 }
The orbit containing 8 is { 7, 8, 9 }
The orbit containing 9 is { 7, 8, 9 }
> for i in [1..9] do
for> "The order of the stabilizer of",i,"is",#Stabilizer(G,i);
for> end for;
The order of the stabilizer of 1 is 18
The order of the stabilizer of 2 is 18
The order of the stabilizer of 3 is 18
The order of the stabilizer of 4 is 18
The order of the stabilizer of 5 is 36
The order of the stabilizer of 6 is 36
The order of the stabilizer of 7 is 24
The order of the stabilizer of 8 is 24
The order of the stabilizer of 9 is 24
> for i in [1..9] do
for> #(i^G) * #Stabilizer(G,i);
for> end for;
72
72
72
72
72
72
72
72
72
72
```

If i is kept fixed and we calculate i^g as g ranges over all elements of G , then the numbers i^g that arise are the elements of the orbit containing i . Each element of the orbit occurs the same number of times. The number of $g \in G$ such that $i^g = i$ is (by definition) the order of the stabilizer of i . For each j in the orbit containing i , the elements of g such that $i^g = j$ form a right coset of the stabilizer of i ; the number of elements in each coset is the same as the number of elements in the stabilizer. So the number of elements in the orbit multiplied by the order of the stabilizer equals the total number of elements of G .

```
> H:=Stabilizer(G,7);
> #H;
24
```

