THE UNIVERSITY OF SYDNEY

MATH2902 Vector Spaces

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Tutorial 7

1. Let $A \in \mathrm{Mat}(n \times n, \mathbb{R})$, and suppose that the columns of A form an orthonormal basis of $\mathbb{R}^n$. Show that ${}^tA = A^{-1}$, and deduce that the rows of A form an orthonormal basis of ${}^t\mathbb{R}^n$.

Solution.

The (i,j)-entry of $({}^tA)A$ is obtained by multiplying the i^{th} row of ${}^tA$ by the j^{th} column of A. But the i^{th} row of ${}^tA$ is just the transpose of the i^{th} column of A; so this is just the dot product of the i^{th} and j^{th} columns of A. Since the columns of A form an orthonormal set this shows that the (i,j)-entry of $({}^tA)A$ is 0 if $i \neq j$ and 1 if $i = j$. So $({}^tA)A = I$. By Theorem 2.9 of the text it follows that ${}^tA = A^{-1}$.

From the above we know that $A({}^tA) = I$. But the (i,j)-entry of $A({}^tA)$ is the dot product of the i^{th} and j^{th} rows of A, and $A({}^tA) = I$ says that this (i,j)-entry is δ_{ij}. That is, the dot product of the i^{th} and j^{th} rows of A is 0 if $i \neq j$ and 1 if $i = j$, as required.

2. Let V be an inner product space and U a subspace of V. Define

$$U^\perp = \{\, v \in V \mid \langle u, v\rangle = 0 \text{ for all } u \in U \,\}.$$

(*i*) Use Theorem 3.10 to prove that $U^\perp$ is a subspace of V.

(*ii*) Prove that if x, $x' \in U$ and y, $y' \in U^\perp$ and $x + y = x' + y'$ then $x = x'$ and $y = y'$.

Solution.

(*i*) The zero element $\underset{\sim}{0}$ of the space V is certainly an element of $U^\perp$, since $\langle u, 0\rangle = 0$ for all $u \in U$. (See 5.1.1, p. 100.) So $U^\perp$ is nonempty.

Let v, $w \in U^\perp$ and let λ be a scalar. For all $u \in U$ we have that $\langle u, v\rangle = \langle u, w\rangle = 0$, and now linearity of $\langle\ ,\ \rangle$ in the second variable yields

$$\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle = 0 + 0 = 0$$

and

$$\langle u, \lambda v\rangle = \lambda\langle u, v\rangle = \lambda 0 = 0$$

for all $u \in U$, showing that $v+w$ and λv are both in $U^\perp$. Since v, w and λ were arbitrary this shows that $U^\perp$ is closed under addition and scalar multiplication. By Theorem 3.10 it follows that $U^\perp$ is a subspace.

(*ii*) Suppose that x, $x' \in U$ and y, $y' \in U^\perp$ and $x + y = x' + y'$. Then $x - x' = y' - y$, and

$$\begin{aligned}\langle x - x', x - x'\rangle &= \langle x - x', y - y'\rangle \\ &= \langle x, y\rangle - \langle x, y'\rangle - \langle x', y\rangle + \langle x', y'\rangle \\ &= 0\end{aligned}$$

since y and y', being in $U^\perp$, must be orthogonal to x and x' (which are in U). By positive definiteness of the inner product we conclude that $x - x' = \underset{\sim}{0}$, and therefore $x = x'$. Since $y - y' = x - x' = 0$ it also follows that $y = y'$.

If U is a finitely generated subspace of the inner product space V then there exists a function $P\colon V \to U$ (the orthogonal projection) such that $v - P(v) \in U^\perp$ for all $v \in V$. Hence in this case each $v \in V$ can be expressed in the form $x + y$ with $x \in U$ and $y \in U^\perp$, by putting $x = P(v)$ and $y = v - P(v)$. By 2 (*ii*) above this expression is unique. (Note that these results need not apply if U is not finitely generated.)

3. Let V be a finite dimensional inner product space and U a subspace of V. Suppose that $x_1, x_2, \ldots, x_n$ form an orthogonal basis of U and $y_1, y_2, \ldots, y_m$ form an orthogonal basis of $U^\perp$. Prove that $x_1, \ldots, x_n, y_1, \ldots, y_m$ form an orthogonal basis of V. Hence prove that the sum of the dimensions of U and $U^\perp$ equals the dimension of V.

Solution.

Let $v \in V$. By the comments above there exist $x \in U$ and $y \in U^\perp$ with $v = x + y$. Since the x_i span U and the y_j span $U^\perp$ there exist scalars λ_i and μ_j with $x = \sum_{i=1}^n \lambda_i x_i$ and $y = \sum_{j=1}^m \mu_j y_j$. Hence

$$v = x + y = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n + \mu_1 y_1 + \mu_2 y_2 + \cdots + \mu_m y_m,$$

and since v was arbitrary this shows that the x_i and y_j together span V. To prove that they form an orthogonal basis it remains to prove that they are all nonzero and that the inner product of any one of them with any other is zero. (Linear independence follows from this by Proposition 5.4.)

We are give that the x_i form an orthogonal basis of U; so each x_i is nonzero and $\langle x_i, x_k\rangle = 0$ if $i \neq k$. Similarly, since the y_j form an orthogonal basis of $U^\perp$, each y_j is nonzero and $\langle y_j, y_l\rangle = 0$ if $j \neq l$. Finally, $\langle x_i, y_j\rangle = 0$ for all i and j, since $x_i \in U$ and $y_j \in U^\perp$, and (by definition) $\langle x, y\rangle = 0$ for all $x \in U$ and $y \in U^\perp$.

4. Let U be the subspace of ${}^t\mathbb{R}^3$ spanned by $(1,1,1)$ and $(1,1,-2)$. Find a basis for $U^\perp$.

Solution.

The two given vectors are orthogonal to each other, and therefore form an orthogonal basis of U. Since ${}^t\mathbb{R}^3$ has dimension 3 we deduce from Exercise 3 that the dimension of $U^\perp$ is 1. So any nonzero vector which is orthogonal to both the given vectors will be a (one-element) basis of $U^\perp$. It is fairly obvious that $(1,-1,0)$ is such a vector. (This was found by solving the simultaneous equations $(1,1,1)\cdot(x,y,z) = (1,1,-2)\cdot(x,y,z) = 0$. Alternatively, one could choose any $v \in {}^t\mathbb{R}^3$ which is not in U and apply the Gram-Schmidt to the basis $((1,1,1),(1,1,-2),v)$ to obtain an orthogonal basis of ${}^t\mathbb{R}^3$. For instance, choosing $v = (1,0,0)$ gives

$$(1,0,0) - \frac{(1,1,1)\cdot(1,0,0)}{(1,1,1)\cdot(1,1,1)}(1,1,1) - \frac{(1,1,-2)\cdot(1,0,0)}{(1,1,-2)\cdot(1,1,-2)}(1,1,-2) = (\tfrac{1}{2},-\tfrac{1}{2},0)$$

as the basis element of $U^\perp$.)

5. Let $A \in \mathrm{Mat}(m \times n, \mathbb{R})$. Show that $x \in \mathbb{R}^n$ is a solution of the equations $Ax = 0$ if and only if ${}^t x$ is orthogonal to each of the rows of A. Deduce that the dimension of the solution space of $Ax = 0$ equals the dimension of the orthogonal complement of the row space of A.

Solution.

The i^{th} entry of Ax is $a_i x = a_i \cdot ({}^t x)$, where a_i is the i^{th} row of A. So $Ax = 0$ if and only if $a_i \cdot ({}^t x) = 0$ for all i.

If $a_i \cdot ({}^t x) = 0$ for all i then $(\sum_{i=1}^m \lambda_i a_i)\cdot({}^t x) = 0$ for any choice of the scalars λ_i, and so ${}^t x$ is orthogonal to all elements of the row space of A. Conversely, if ${}^t x$ is orthogonal to everything in the rowspace of A then it is certainly orthogonal to each of the rows. So the solution space of $Ax = 0$ is exactly the orthogonal complement of the row space of A, except that the vectors are written as columns instead of rows. This notational change obviously does not change the dimension. Note also (by Exercise 3) that if r is the dimension of the row space of A then the dimension of the solution space will be $m - r$.

6. Find an orthonormal basis for the 1-eigenspace of $\begin{pmatrix} 2&1&2&2\\ 1&2&2&2\\ 2&2&5&4\\ 2&2&4&5 \end{pmatrix}$. Find also an orthonormal basis for the orthogonal complement of this space, and verify that this orthogonal complement equals the 11-eigenspace of the above matrix. Using the elements of these bases as the columns, construct a matrix T such that ${}^t T = T^{-1}$ and $T^{-1}AT = \mathrm{diag}(1,1,1,11)$.

Solution.

The 1-eigenspace of A is the solution space of $Av = v$, or (equivalently) $(A - I)v = 0$. Applying row operations to $A - I$ yields the reduced echelon matrix with $(1,1,2,2)$ as first row and all other rows zero. The last three variables of v are all free, and the general solution is

$$\begin{pmatrix} x\\ y\\ z\\ w \end{pmatrix} = \begin{pmatrix} -\alpha-2\beta-2\gamma\\ \alpha\\ \beta\\ \gamma \end{pmatrix} = \alpha\begin{pmatrix} -1\\ 1\\ 0\\ 0 \end{pmatrix} + \beta\begin{pmatrix} -2\\ 0\\ 1\\ 0 \end{pmatrix} + \gamma\begin{pmatrix} -2\\ 0\\ 0\\ 1 \end{pmatrix}.$$

The three columns appearing in the right hand side of the above expression form a basis of the 1-eigenspace of A, but not an orthogonal basis. Applying the Gram-Schmidt process replaces the second basis vector by

$$\begin{pmatrix} -2\\ 0\\ 1\\ 0 \end{pmatrix} - \frac{(-1)(-2)}{(-1)^2+1^2}\begin{pmatrix} -1\\ 1\\ 0\\ 0 \end{pmatrix} = \begin{pmatrix} -1\\ -1\\ 1\\ 0 \end{pmatrix}$$

and the third by

$$\begin{pmatrix} -2\\ 0\\ 0\\ 1 \end{pmatrix} - \begin{pmatrix} -1\\ 1\\ 0\\ 0 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} -1\\ -1\\ 1\\ 0 \end{pmatrix} = \begin{pmatrix} -1/3\\ -1/3\\ -2/3\\ 1 \end{pmatrix}.$$

Dividing each of these by its length (to normalize) give the orthonormal basis

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -1\\ 1\\ 0\\ 0 \end{pmatrix},\quad \frac{1}{\sqrt{3}}\begin{pmatrix} -1\\ -1\\ 1\\ 0 \end{pmatrix},\quad \frac{1}{\sqrt{15}}\begin{pmatrix} -1\\ -1\\ 2\\ 3 \end{pmatrix}.$$

By Exercise 5 we know that this 1-eigenspace is, in effect, the orthogonal complement of the row space of $A - I$, which we have shown to be spanned by $(1,1,2,2)$. Hence ${}^t(1,1,2,2)$ spans the orthogonal complement of the 1-eigenspace of A. The fact that this column is also an eigenvector for A is surprising at first, but ceases to be surprising in light of the fact, shown in Exercise 4 of Tutorial 2, that all eigenvectors belonging to eigenvectors other than 1 must be orthogonal to all the 1-eigenvectors. (This is the important property of real symmetric matrices: eigenspaces corresponding to distinct eigenvalues are orthogonal to each other.) It is trivial to verify that if $v = {}^t(1,1,2,2)$ then $Av = 11v$. We are asked for an orthonormal basis of the 11-eigenspace; so we must replace v by $\frac{1}{\sqrt{10}}v$ (since $||v|| = \sqrt{10}$).

Define

$$T = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{15} & 1/\sqrt{10}\\ 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{15} & 1/\sqrt{10}\\ 0 & 1/\sqrt{3} & -2/\sqrt{15} & 2/\sqrt{10}\\ 0 & 0/\sqrt{3} & 3/\sqrt{15} & 2/\sqrt{10} \end{pmatrix}.$$

Exercise 1 shows that ${}^t T = T^{-1}$, and since the columns of T are eigenvectors of A for the eigenvalues 1, 1, 1 and 11 (respectively) we know that

$$T^{-1}AT = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&11 \end{pmatrix}.$$

(See §2f#6, and see also Proposition 9.4.)