THE UNIVERSITY OF SYDNEY

MATH2902 Vector Spaces

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2902/)

Semester1, 2001 — Lecturer: R. Howlett

Tutorial 8

1. Use $\det(AB) = \det A \det B$ and $\det {}^tA = \det A$ to prove that the determinant of a real orthogonal matrix must be ± 1. (A 3×3 real orthogonal matrix corresponds to a rotation of the coordinate axes if its determinant is 1; orthogonal matrices of determinant -1 change right-handed coordinate systems into left-handed ones.)

Solution.

$1 = \det I = \det({}^tAA) = \det {}^tA \det A = (\det A)^2$, and so $\det A = \pm 1$.

2. Find a rotation of the coordinate axes which changes the equation of the given quadric surface to the form $a(x')^2 + b(y')^2 + c(z')^2 = \text{constant}$.

(*i*) $6x^2 + 4y^2 - 4z^2 + 2xy - 6xz + 2yz = 140$

(*ii*) $4x^2 - 14y^2 + 12z^2 - 2xy - 2xz - 10yz = -780$

(*iii*) $4x^2 + 12y^2 + 2z^2 + 2xy + 2xz + 6yz = 104$

Solution.

(*i*) The equation can be written as ${}^t\underset{\sim}{x} A \underset{\sim}{x} = 140$, where

$$\underset{\sim}{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } A = \begin{pmatrix} 6 & 1 & -3 \\ 1 & 4 & 1 \\ -3 & 1 & -4 \end{pmatrix}.$$

A rotation of coordinate axes is a change of variable of the form $\underset{\sim}{x} = P\underset{\sim}{x}'$, where P is an orthogonal matrix of determinant 1, and we need to choose P so that ${}^tPAP$ is diagonal.

The first step is to find the eigenvalues and corresponding eigenspaces of the matrix A. The characteristic polynomial of A (the determinant of $A - xI$) is

$$\begin{aligned}(6-x)((4-x)(-4-x)-1) - ((-4-x)+3) - 3(1+3(4-x)) &= (6-x)(x^2-17) + x + 1 + 9x - 39 \\ &= -(x^3 - 6x^2 - 27x + 140) \\ &= -(x-7)(x+5)(x-4)\end{aligned}$$

so that the eigenvalues are 7, -5 and 4.

To find the eigenspace corresponding to the eigenvalue 7 we must solve the equations $(A - 7I)\underset{\sim}{x} = \underset{\sim}{0}$. Applying the pivoting algorithm (row operations) to $A - 7I$ gives

$$\begin{pmatrix} -1 & 1 & -3 \\ 1 & -3 & 1 \\ -3 & 1 & -11 \end{pmatrix} \xrightarrow[R_1:=-R_1]{\substack{R_2:=R_2+R_1 \\ R_3:=R_3-3R_1}} \begin{pmatrix} 1 & -1 & 3 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{pmatrix} \xrightarrow[R_1:=R_1+R_2]{\substack{R_3:=R_3-R_2 \\ R_2:=(-1/2)R_2}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and it follows that the column ${}^t(-4,-1,1)$ spans the eigenspace. Similarly, row operations applied to the matrices $A + 5I$ and $A - 4I$ give the reduced echelon matrices

$$\begin{pmatrix} 1 & 0 & -2/7 \\ 0 & 1 & 1/7 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{pmatrix}$$

respectively, and we see that ${}^t(2,-1,7)$ and ${}^t(-1,5,1)$ span the corresponding eigenspaces. The theory tells us that the eigenspaces must be orthogonal to each other relative to the dot product on $\mathbb{R}^3$ (since A is symmetric), and it is advisable (and quick) to check this at this point. For instance,

$$\begin{pmatrix} -4 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 7 \end{pmatrix} = (-4) \times 2 + (-1) \times (-1) + 1 \times 7 = -8 + 1 + 7 = 0.$$

Choose a unit vector in each of the eigenspaces and let P be the matrix with these unit vectors as its columns. Then P will be orthogonal. Its determinant will be either 1 or -1; if it turns out to be -1 simply replacing one of the columns by its negative will change the determinant to 1, thereby ensuring that P is a rotation matrix. There are exactly 24 suitable matrices P, one of which is

$$P = \begin{pmatrix} -4/\sqrt{18} & 2/\sqrt{54} & -1/\sqrt{27} \\ -1/\sqrt{18} & -1/\sqrt{54} & 5/\sqrt{27} \\ 1/\sqrt{18} & 7/\sqrt{54} & 1/\sqrt{27} \end{pmatrix}.$$

(The other possibilities are obtainable by writing the columns down in some other order and/or changing the signs of some of the columns.) Our choice of P converts the equation to $7(x')^2 - 5(y')^2 + 4(z')^2 = 140$. Such a surface is known as a "hyperboloid of one sheet". The intersection of our surface with any of the planes $y' =$ constant (that is, planes parallel to the $x'z'$-plane) is an ellipse $7(x')^2 + 4(z')^2 =$ constant whose size increases rapidly as y' goes to $\pm\infty$. The planes $x' =$ constant and $z' =$ constant intersect the surface in hyperbolas. The effect is somewhat like rotating the hyperbola $7(x')^2 - 5(y')^2 = 140$ about the y'-axis, although the "rotation" is elliptical rather than circular. (More exactly, rotate $X^2 - Y^2 = 1$ about the Y-axis, to obtain the surface $X^2 - Y^2 + Z^2 = 1$, then stretch the coordinate axes by putting $x' = \sqrt{20}X$, $y' = \sqrt{28}Y$ and $z' = \sqrt{35}Z$.)

(*ii*) The calculations are totally analogous to those in the first part. The characteristic polynomial is

$$\begin{aligned} (4-x)((-14-x)(12-x)-25) &+ (-(12-x)-5) - (5+(-14-x)) \\ &= (4-x)(x^2+2x-193) + x - 17 + x + 9 \\ &= (4-x)(x^2+2x-195) \\ &= -(x-4)(x+15)(x-13) \end{aligned}$$

giving eigenvalues of 4, -15 and 13. Applying row operations to $A - 4I$, $A + 15I$ and $A - 13I$ one easily obtains the reduced echelon matrices

$$\begin{pmatrix} 1 & 0 & -13 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -16/3 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 1/11 \\ 0 & 1 & 2/11 \\ 0 & 0 & 0 \end{pmatrix}$$

respectively. The following matrix P has determinant 1 and unit vectors from the three eigenspaces as its columns:

$$P = \begin{pmatrix} 13/\sqrt{171} & 1/\sqrt{266} & -1/\sqrt{126} \\ -1/\sqrt{171} & 16/\sqrt{266} & -2/\sqrt{126} \\ 1/\sqrt{171} & 3/\sqrt{266} & 11/\sqrt{126} \end{pmatrix}.$$

Putting $\underset{\sim}{x} = P\underset{\sim}{x}'$ gives the equation $4(x')^2 - 15(y')^2 + 13(z')^2 = -780$. The surface is a hyperboloid of two sheets, obtained by rotating the hyperbola $X^2 - Y^2 = -1$ about the Y-axis and then stretching the axes. ("Two sheets" because the surface has two parts which are not connected to each other, on opposite sides of the plane $Y = 0$.)

(*iii*) This time the characteristic polynomial is

$$\begin{aligned} (4-x)((12-x)(2-x)-9) - ((2-x)-3) &+ (3-(12-x)) \\ &= (4-x)(x^2-14x+15) + x + 1 + x - 9 \\ &= (4-x)(x^2-14x+13) \\ &= -(x-4)(x-1)(x-13) \end{aligned}$$

so that the eigenvalues are 4, 1 and 13. Row operations applied to $A - 4I$, $A - I$ and $A - 13I$ yield the reduced echelon matrices

$$\begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1/4 \\ 0 & 1 & 1/4 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -7/2 \\ 0 & 0 & 0 \end{pmatrix},$$

and consequently we find that a suitable rotation matrix is

$$P = \begin{pmatrix} 5/\sqrt{27} & 1/\sqrt{18} & 1/\sqrt{54} \\ -1/\sqrt{27} & 1/\sqrt{18} & 7/\sqrt{54} \\ 1/\sqrt{27} & -4/\sqrt{18} & 2/\sqrt{54} \end{pmatrix}$$

The equation of the surface becomes $4(x')^2 + (y')^2 + 13(z')^2 = 104$, and we see that it is an ellipsoid (like a severely maltreated sphere).

3. A square complex matrix A is said to be *normal* if it commutes with A^*. (That is, $AA^* = A^*A$. Here $A^* \stackrel{\text{def}}{=} {}^t\overline{A}$.) Prove that if A is normal and U is unitary then U^*AU is normal.

Solution.

Since the transpose conjugate operation $*$ reverses products we see that $(U^*AU)^* = U^*A^*(U^*)^* = U^*A^*U$. Since U is unitary we have that $UU^* = I$, and now

$$\begin{aligned} (U^*AU)^*(U^*AU) &= U^*A^*UU^*AU = U^*A^*AU \\ &= U^*AA^*U = U^*AUU^*A^*U = (U^*AU)(U^*AU)^*, \end{aligned}$$

showing that U^*AU is normal.

4. Let A be a complex $n \times n$ matrix and suppose that there exists a unitary matrix U such that U^*AU is diagonal. Prove that $A(A^*) = (A^*)A$.
(Hint: Let $D = U^*AU$, and prove first that $D(D^*) = (D^*)D$.)

Solution.

It is trivial that $D_1D_2 = D_2D_1$ if D_1 and D_2 are both diagonal matrices. If D is diagonal then so is D^*, whence we deduce that $DD^* = D^*D$. Now since $U^* = U^{-1}$ the equation $D = U^*AU$ gives $A = UDU^*$, and (as in Exercise 3)

$$\begin{aligned} A^*A &= (UDU^*)^*(UDU^*) = UD^*U^*UDU^* = UD^*DU^* \\ &= UDD^*U^* = UDU^*UD^*U^* = (UDU^*)(UDU^*)^* = AA^*. \end{aligned}$$

5. (*i*) Suppose that $A \in \mathrm{Mat}(n \times n, \mathbb{C})$ is normal and upper triangular. Prove that A is diagonal.
(Hint: 'Upper triangular' means $A_{ij} = 0$ for $i > j$. Prove that the $(1,1)$-entry of $A(A^*)$ is $\sum_{i=1}^n |A_{1j}|^2$ whereas the $(1,1)$-entry of $(A^*)A$ is $|A_{11}|^2$, and deduce that $A_{1j} = 0$ for all $j > 1$. Then consider the $(2,2)$-entries of $A(A^*)$ and $(A^*)A$, then $(3,3)$, and so on.)

(*ii*) It can be shown that for any $A \in \mathrm{Mat}(n \times n, \mathbb{C})$ there exists a unitary matrix U such that U^*AU is upper triangular. (The proof of this is very similar to the proof of Theorem 5.19.) Use this fact together with Exercise 3 and Part (*i*) to prove that for every normal matrix A there exists a unitary U with U^*AU diagonal.

Solution.

(*i*) We use induction on i to prove

($) $\qquad A_{ij} = 0$ for all $j > i$.

Let us use the notation X_{rs} for the (r,s)-entry of a matrix X. Then $(1,1)$-entry of AA^* is given by

$$(AA^*)_{11} = \sum_{j=1}^n A_{1j}(A^*)_{j1} = \sum_{j=1}^n A_{1j}\overline{A_{1j}} = \sum_{j=1}^n |A_{1j}|^2$$

while the $(1,1)$ entry of A^*A is

$$(A^*A)_{11} = \sum_{i=1}^n (A^*)_{1i}A_{i1} = \sum_{i=1}^n \overline{A}_{i1}A_{i1} = \sum_{i=1}^n |A_{i1}|^2.$$

Since $A_{i1} = 0$ for $i > 1$ it follows that $(A^*A)_{11} = |A_{11}|^2$. But $(A^*A)_{11} = (AA^*)_{11}$ (since A is normal), and so

$$0 = (AA^*)_{11} - (A^*A)_{11} = \Big(\sum_{j=1}^n |A_{1j}|^2\Big) - |A_{11}|^2 = \sum_{j>1} |A_{1j}|^2.$$

Since each $|A_{ij}|^2$ is real and nonnegative, the only way that this sum can be zero is if each term is zero. So $A_{1j} = 0$ for all $j > 1$, proving that ($) is satisfied in the case $i = 1$.

Let $k > 1$ and assume that ($) is satisfied for all $i < k$. In particular, putting $j = k$ in ($) this gives $A_{ik} = 0$ for all $i < k$. We also have that $A_{ik} = 0$ for all $i > k$ since A is upper triangular. So $A_{ik} = 0$ for $i \neq k$, and

$$(A^*A)_{kk} = \sum_{i=1}^n (A^*)_{ki}A_{ik} = \sum_{i=1}^n \overline{A}_{ik}A_{ik} = |A_{kk}|^2.$$

Furthermore, $A_{kj} = 0$ for all $j < k$ (since A is upper triangular), and so

$$\begin{aligned} (AA^*)_{kk} &= \sum_{j=1}^n A_{kj}(A^*)_{jk} = \sum_{j=1}^n A_{kj}\overline{A_{kj}} \\ &= \sum_{j=1}^n |A_{kj}|^2 = \sum_{j=k}^n |A_{kj}|^2. \end{aligned}$$

Normality of A gives $(A^*A)_{kk} = (AA^*)_{kk}$; therefore

$$0 = (AA^*)_{kk} - (A^*A)_{kk} = \Big(\sum_{j=k}^n |A_{kj}|^2\Big) - |A_{kk}|^2 = \sum_{j>k} |A_{kj}|^2,$$

and this forces $A_{kj} = 0$ for all $j > k$. So ($) holds for $i = k$, and our induction is complete.

So ($) holds for all i, whence A is lower triangular as well as upper triangular. So A is diagonal.

(*ii*) Let A be normal and choose a unitary U such that $T = U^*AU$ is upper triangular. Exercise 3 says that T is normal; so by Part (*i*) it must be diagonal.