The University of Sydney

MATH2902 Vector Spaces

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2902/)

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Tutorial 7

1. Let $A \in \mathrm{Mat}(n \times n, \mathbb{R})$, and suppose that the columns of A form an orthonormal basis of $\mathbb{R}^n$. Show that ${}^{t}A = A^{-1}$, and deduce that the rows of A form an orthonormal basis of ${}^{t}\mathbb{R}^n$.

2. Let V be an inner product space and U a subspace of V. Define

$$U^{\perp} = \{\, v \in V \mid \langle u, v\rangle = 0 \text{ for all } u \in U \,\}.$$

(*i*) Use Theorem 3.10 to prove that $U^{\perp}$ is a subspace of V.

(*ii*) Prove that if x, $x' \in U$ and y, $y' \in U^{\perp}$ and $x + y = x' + y'$ then $x = x'$ and $y = y'$.

If U is a finitely generated subspace of the inner product space V then there exists a function $P\colon V \to U$ (the orthogonal projection) such that $v - P(v) \in U^{\perp}$ for all $v \in V$. Hence in this case each $v \in V$ can be expressed in the form $x+y$ with $x \in U$ and $y \in U^{\perp}$, by putting $x = P(v)$ and $y = v - P(v)$. By 2 (*ii*) above this expression is unique. (Note that these results need not apply if U is not finitely generated.)

3. Let V be a finite dimensional inner product space and U a subspace of V. Suppose that $x_1, x_2, \ldots, x_n$ form an orthogonal basis of U and $y_1, y_2, \ldots, y_m$ form an orthogonal basis of $U^{\perp}$. Prove that $x_1, \ldots, x_n, y_1, \ldots, y_m$ form an orthogonal basis of V. Hence prove that the sum of the dimensions of U and $U^{\perp}$ equals the dimension of V.

4. Let U be the subspace of ${}^{t}\mathbb{R}^3$ spanned by $(1, 1, 1)$ and $(1, 1, -2)$. Find a basis for $U^{\perp}$.

5. Let $A \in \mathrm{Mat}(m \times n, \mathbb{R})$. Show that $x \in \mathbb{R}^n$ is a solution of the equations $Ax = 0$ if and only if ${}^{t}x$ is orthogonal to each of the rows of A. Deduce that the dimension of the solution space of $Ax = 0$ equals the dimension of the orthogonal complement of the row space of A.

6. Find an orthonormal basis for the 1-eigenspace of $\begin{pmatrix} 2 & 1 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 2 & 2 & 5 & 4 \\ 2 & 2 & 4 & 5 \end{pmatrix}$. Find also an orthonormal basis for the orthogonal complement of this space, and verify that this orthogonal complement equals the 11-eigenspace of the above matrix. Using the elements of these bases as the columns, construct a matrix T such that ${}^{t}T = T^{-1}$ and $T^{-1}AT = \mathrm{diag}(1, 1, 1, 11)$.