THE UNIVERSITY OF SYDNEY

MATH2902 Vector Spaces

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2902/)

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Tutorial 8

1. Use $\det(AB) = \det A \det B$ and $\det {}^{t}A = \det A$ to prove that the determinant of a real orthogonal matrix must be ± 1. (A 3×3 real orthogonal matrix corresponds to a rotation of the coordinate axes if its determinant is 1; orthogonal matrices of determinant -1 change right-handed coordinate systems into left-handed ones.)

2. Find a rotation of the coordinate axes which changes the equation of the given quadric surface to the form $a(x')^2 + b(y')^2 + c(z')^2 = \text{constant}$.

(i) $6x^2 + 4y^2 - 4z^2 + 2xy - 6xz + 2yz = 140$

(ii) $4x^2 - 14y^2 + 12z^2 - 2xy - 2xz - 10yz = -780$

(iii) $4x^2 + 12y^2 + 2z^2 + 2xy + 2xz + 6yz = 104$

3. A square complex matrix A is said to be *normal* if it commutes with A^*. (That is, $AA^* = A^*A$. Here $A^* \stackrel{\text{def}}{=} {}^{t}\overline{A}$.) Prove that if A is normal and U is unitary then U^*AU is normal.

4. Let A be a complex $n \times n$ matrix and suppose that there exists a unitary matrix U such that U^*AU is diagonal. Prove that $A(A^*) = (A^*)A$.
(Hint: Let $D = U^*AU$, and prove first that $D(D^*) = (D^*)D$.)

5. *(i)* Suppose that $A \in \text{Mat}(n \times n, \mathbb{C})$ is normal and upper triangular. Prove that A is diagonal.
(Hint: 'Upper triangular' means $A_{ij} = 0$ for $i > j$. Prove that the $(1,1)$-entry of $A(A^*)$ is $\sum_{i=1}^{n} |A_{1j}|^2$ whereas the $(1,1)$-entry of $(A^*)A$ is $|A_{11}|^2$, and deduce that $A_{1j} = 0$ for all $j > 1$. Then consider the $(2,2)$-entries of $A(A^*)$ and $(A^*)A$, then $(3,3)$, and so on.)

(ii) It can be shown that for any $A \in \text{Mat}(n \times n, \mathbb{C})$ there exists a unitary matrix U such that U^*AU is upper triangular. (The proof of this is very similar to the proof of Theorem 5.19.) Use this fact together with Exercise 3 and Part *(i)* to prove that for every normal matrix A there exists a unitary U with U^*AU diagonal.