THE UNIVERSITY OF SYDNEY

MATH2902 Vector Spaces

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2902/)

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Tutorial 10

1. Prove that isomorphic vector spaces have the same dimension.
(Hint: Use Theorem 4.17. This was proved in Exercise 5 of Tutorial 4.)

2. Is it possible to find subspaces U, V and W of $\mathbb{R}^4$ such that

$$\mathbb{R}^4 = U \oplus V = V \oplus W = W \oplus U\,?$$

3. (*i*) Let V and W be vector spaces over F. Show that the Cartesian product of V and W (see §1b) becomes a vector space if addition and scalar multiplication are defined in the natural way. (This space is called the *external direct sum* of V and W, and is sometimes denoted by '$V \dotplus W$'.)

(*ii*) Show that $V' = \{\,(v,0) \mid v \in V\,\}$ and $W' = \{\,(0,w) \mid w \in W\,\}$ are subspaces of $V \dotplus W$ with $V' \cong V$ and $W' \cong W$, and that $V \dotplus W = V' \oplus W'$.

(*iii*) Prove that $\dim(V \dotplus W) = \dim V + \dim W$.

4. Let S and T be subspaces of a vector space V and let U be a subspace of T such that $T = (S \cap T) \oplus U$. Prove that $S + T = S \oplus U$ (see Tutorial 3 for the definition of $S + T$), and hence deduce that

$$\dim(S + T) = \dim S + \dim T - \dim(S \cap T).$$

5. (*i*) Let S and T be subspaces of a vector space V. Prove that $(s,t) \mapsto s + t$ defines a linear transformation from $S \dotplus T$ to V which has image $S + T$ and kernel isomorphic to $S \cap T$.

(*ii*) The Main Theorem on Linear Transformations (see p. 158 of the book) asserts that if V is a finitely generated vector space and θ a linear transformation from V to another space W, then the sum of the dimensions of $\ker\theta$ and $\operatorname{im}\theta$ equals the dimension of V. Use this and Part (*i*) to give another proof that $\dim(S + T) + \dim(S \cap T) = \dim S + \dim T$.