The University of Sydney

Pure Mathematics 3901

Metric Spaces 2000

Assignment 1

1. Let (X_1, d_1) and (X_2, d_2) be metric spaces, and let $X = X_1 \times X_2$, the Cartesian product of X_1 and X_2. Define $d\colon X \times X \to \mathbb{R}$ by the formula

$$d(x, y) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

for all $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in X.

Prove that d is a metric on X.

Solution.

Let a, $b \in X$. Then $a = (a_1, a_2)$ and $b = (b_1, b_2)$ for some a_1, $b_1 \in X_1$ and a_2, $b_2 \in X_2$, and since d_1 is a metric on X_1 and d_2 is a metric on X_2 we have $d_1(a_1, b_1) = d_1(b_1, a_1) \geq 0$ and $d_2(a_2, b_2) = d_2(b_2, a_2) \geq 0$ (by the condition (M1) in the definition of a metric). Hence

$$\begin{aligned} d(a, b) = d((a_1, a_2), (b_1, b_2)) &= \max\{d_1(a_1, b_1), d_2(a_2, b_2)\} \\ &= \max\{d_1(b_1, a_1), d_2(b_2, a_2)\} = d((b_1, b_2), (a_1, a_2)) = d(b, a). \end{aligned}$$

Furthermore, $d(a, b) \geq d_1(a_1, b_1) \geq 0$. Since a and b were arbitrary points of X we have shown that $d(a, b) = d(b, a) \geq 0$ for all a, $b \in X$. Thus d satisfies (M1).

Suppose that a, $b \in X$ satisfy $d(a, b) = 0$. Writing $a = (a_1, a_2)$ and $b = (b_1, b_2)$ as above, we have

$$0 \leq d_1(a_1, b_1) \leq \max\{d_1(b_1, a_1), d_2(b_2, a_2)\} = d(a, b) = 0,$$

and so $d_1(a_1, b_1) = 0$. Similarly $d_2(a_2, b_2) = 0$. Now since d_1 and d_2 satisfy condition (M2) it follows that $a_1 = b_1$ and $a_2 = b_2$, and therefore

$$a = (a_1, a_2) = (b_1, b_2) = b.$$

Since a, $b \in X$ were arbitrary subject to $d(a, b) = 0$, we have shown that for all a, $b \in X$, if $d(a, b) = 0$ then $a = b$. Conversely, if $a = b$ then $a_1 = b_1$ and

$a_2 = b_2$, and since d_1 and d_2 are metrics it follows that $d_1(a_1, b_1) = 0$ and $d_2(a_2, b_2) = 0$, whence

$$d(a, b) = \max\{d_1(b_1, a_1), d_2(b_2, a_2)\} = \max\{0\} = 0.$$

So $d(a, b) = 0$ if and only if $a = b$, and so d satisfies (M2).

Now let a, b, $c \in X$ be arbitrary, and write a_1, b_1, c_1 for their X_1 components and a_2, b_2, c_2 for their X_2 components. Since d_1 is a metric, (M3) gives

$$d_1(b_1, c_1) \leq d_1(a_1, b_1) + d_1(a_1, c_1) \leq d(a, b) + d(a, c), \tag{1}$$

since $d_1(a_1, b_1) \leq \max\{d_1(a_1, b_1), d_2(a_2, b_2)\} = d(a, b)$ and $d_1(a_1, c_1) \leq d(a, c)$ similarly. In the same way, since d_2 is a metric,

$$d_2(b_2, c_2) \leq d_2(a_2, b_2) + d_2(a_2, c_2) \leq d(a, b) + d(a, c), \tag{2}$$

and (1) and (2) together give

$$d(b, c) = \max\{d_1(b_1, c_1), d_2(b_2, c_2)\} \leq d(a, b) + d(a, c).$$

This holds for all a, b, $c \in X$, and so d satisfies (M3). Since it satisfies all of (M1), (M2) and (M3), it is a metric.

2. Let $\mathcal{U}$ be the set of all subsets of the set $[0, 1] = \{\, x \in \mathbb{R} \mid 0 \leq x \leq 1 \,\}$, and let $\mathcal{V} = \{\emptyset, [0, 1]\}$, a subset of $\mathcal{U}$. Let $X = ([0, 1], \mathcal{U})$ and $Y = ([0, 1], \mathcal{V})$.

(i) Show that X and Y are both topological spaces.

(ii) Describe all the continuous functions from X to X, all the continuous functions from X to Y, all the continuous functions from Y to X and all the continuous functions from Y to Y.

(Recall that the definition of continuity for functions from one topological space to another is that a function is continuous if and only if the preimage of every open set is open.)

Solution.

(i) Recall that a collection of subsets of a set S is called a *topology* on S if and only if the collection is closed under arbitrary unions and finite intersections, and S and $\emptyset$ are both in the collection. We must show that $\mathcal{U}$ and $\mathcal{V}$ both satisfy these properties (with $S = [0, 1]$).

Since $\mathcal{U}$ consists of all subsets of $[0, 1]$, in particular $[0, 1]$ and $\emptyset$ are in $\mathcal{U}$. The union of any family of sets that are subsets of $[0, 1]$ is obviously a subset of $[0, 1]$, and also the intersection of any family of sets that are subsets of $[0, 1]$ is a subset of $[0, 1]$. So $\mathcal{U}$ is closed under arbitrary unions and arbitrary intersections; hence it is closed under arbitrary unions and finite intersections, as required.

Since by definition $\mathcal{V} = \{\emptyset, [0,1]\}$, there is no doubt that $[0,1]$ and $\emptyset$ are in $\mathcal{V}$. The union of a family of sets, all of which are either $[0,1]$ or $\emptyset$, is clearly $[0,1]$ if one or more of the sets in the family is $[0,1]$, and is $\emptyset$ otherwise; similarly the intersection of such a family is $\emptyset$ if one or more of the sets in the family is $\emptyset$, and is $[0,1]$ otherwise. So $\mathcal{V}$ also is closed under arbitrary unions and intersections.

(*ii*) As explained in lectures, a function from one toplogical space to another means a function between the underlying sets of those spaces. Thus by "a function from X to Y" I mean just a function from $[0,1]$ to $[0,1]$; however, when determining whether or not the function is continuous we need to know that we use the topology $\mathcal{U}$ for $[0,1]$ considered as the domain of f and the topology $\mathcal{V}$ for $[0,1]$ considered as the codomain of f. Thus a continuous function $X \to Y$ means a function $[0,1] \to [0,1]$ such that the preimage of every set in $\mathcal{V}$ is in $\mathcal{U}$.

Observe first that if A and B are any two sets and $f\colon A \to B$ any function then the preimage of B is A and the preimage of the empty subset of B is the empty subset of A. To see this, observe that by definition

$$f^{-1}(B) = \{\, x \in A \mid f(x) \in B \,\} = A,$$

since the fact that f is a function from A to B guarantees that the statement "$f(x) \in B$" is true for all $x \in A$. Similarly,

$$f^{-1}(\emptyset) = \{\, x \in A \mid f(x) \in \emptyset \,\} = \emptyset,$$

since no element x can satisfy the condition "$f(x) \in \emptyset$": it is impossible for $f(x)$ to be an element of $\emptyset$ since $\emptyset$ has no elements.

Now let $A = (S, \mathcal{T})$ be any topological space and f any function from A to Y. Then f is continuous if and only if $f^{-1}(\emptyset) \in \mathcal{T}$ and $f^{-1}([0,1]) \in \mathcal{T}$ (since $\emptyset$ and $[0,1]$ are the only sets in the topology $\mathcal{V}$). But $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}([0,1]) = S$, and certainly $\emptyset \in \mathcal{T}$ and $S \in \mathcal{T}$, whatever the topology $\mathcal{T}$ is, since it is part of the definition of a topology that the empty set and the whole set must always be open. So every function from A to Y is continuous.

Similarly, any function f from X to any topological space A will necessarily be continuous, since all subsets of $[0,1]$ are open sets of X. By definition, f is continuous if and only if $f^{-1}(U) \in \mathcal{U}$ for all open sets U of A. But $f^{-1}(U)$ is a subsets of $[0,1]$, and hence is in $\mathcal{U}$ in every case, since $\mathcal{U}$ consists of all subsets of $[0,1]$. So every f satisfies the requirements for continuity.

The above shows that every function from X to X is continuous, every function from Y to Y is continuous, and every function from X to Y is continuous; we have even given two proofs of this last fact. It remains to determine which functions from Y to X are continuous.

Let us show first that constant functions from Y to X are continuous. So, suppose that $c \in [0,1]$ and $f\colon [0,1] \to [0,1]$ satisfies $f(x) = c$ for all $x \in [0,1]$. Let U be an open subset of the codomain of f. (Since the topology for the codomain is $\mathcal{U}$ in this case, U can be any subset of $[0,1]$.) Now

$$f^{-1}(U) = \{\, x \in [0,1] \mid f(x) \in U \,\} = \{\, x \in [0,1] \mid c \in U \,\} = \begin{cases} \emptyset & \text{if } c \notin U, \\ [0,1] & \text{if } c \in U. \end{cases}$$

In either case (whether $c \in U$ or $c \notin U$) the preimage of U is open, since $\emptyset$ and $[0,1]$ are the two open sets of Y.

Now we show that, conversely, every continuous function from Y to X must be constant. Suppose that f is a continuous function from Y to X, and let $c = f(0)$. Since all subsets of $[0,1]$ are open sets of X, the singleton set $\{c\}$ is open, and since f is continuous it follows that $f^{-1}(\{c\})$ is open. But

$$f^{-1}(\{c\}) = \{\, x \in [0,1] \mid f(x) \in \{c\} \,\} = \{\, x \in [0,1] \mid f(x) = c \,\},$$

and this set is certainly not empty, since $f(0) = c$ shows that $0 \in f^{-1}(\{c\})$. But the only open set of Y that is not empty is the whole set $[0,1]$. So we conclude that

$$\{\, x \in [0,1] \mid f(x) = c \,\} = [0,1],$$

and thus $f(x) = c$ for all $x \in [0,1]$. That is, f is constant, as claimed. So the continuous functions from Y to X are precisely the constant functions.

3. Let d be a metric on the set X. Using results from Tutorials 1 and 2 (which you may quote without proof) show that there exists a metric D on X with the following properties: $D(x,y) \le 1$ for all x, $y \in X$; every open ball of the metric space (X,d) is an open ball of the metric space (X,D); every open ball of the metric space (X,D), excluding X itself, is an open ball of the metric space (X,d).

Solution.

By Question 7 of Tutorial 2, the formula $D(x,y) = d(x,y)/(1 + d(x,y))$ (for all x, $y \in X$) defines a metric on X. We shall not repeat the proof of this here. Everything that we do have to prove follows from the following fact: the formula $f(r) = r/(1+r)$ defines a strictly increasing bijective function $f\colon [0,\infty) \to [0,1)$. So we start by proving this.

If $r \in [0,\infty)$ then $0 \le r < 1+r$, and so $0 \le r/(1+r) < (1+r)/(1+r) = 1$. Thus the given formula does define a function $[0,\infty) \to [0,1)$. Now let r, $s \in [0,\infty)$ with $0 \le r < s$. Then $0 < 1+r < 1+s$; so $1/(1+s) < 1/(1+r)$, giving $f(r) = r/(1+r) = 1 - (1/(1+r)) < 1 - (1/(1+s) = s/(1+s) = f(s)$. This shows that f is strictly increasing, and also one-to-one, on $[0,\infty)$. To show that f is onto we show that if $t \in [0,1)$ then $r = t/(1-t)$ is in $[0,\infty)$ and satisfies $f(r) = t$. Observe that $t \in [0,1)$ gives $1-t > 0$; so $1/(1-t) > 0$, and since $t \geq 0$ we have $t/(1-t) \geq 0/(1-t) = 0$. So $r \in [0,\infty)$, and now $f(r) = r/(1+r) = \frac{t}{1-t}/(1+\frac{t}{1-t}) = t/((1-t)+t) = t$, as required. (This also shows that the inverse function $f^{-1}\colon [0,1) \to [0,\infty)$ is given by the formula $f^{-1}(t) = t/(1-t)$.)

Let x, $y \in X$. Then $D(x,y) = d(x,y)/(1+d(x,y)) = f(d(x,y)) \in [0,1)$. So $D(x,y) < 1$ for all x, $y \in X$.

Let B be any open ball in (X,d). That is, there exist some $a \in X$ and $r > 0$ such that $B = B_d(a,r) = \{x \in X \mid d(a,x) < r\}$. We shall show that B is an open ball in (X,D) by showing that $B_d(a,r) = B_D(a,f(r))$. Now if $x \in B_d(a,r)$ then $d(a,x) < r$, and, since f is strictly increasing, $f(d(a,x)) < f(r)$. But $f(d(a,x)) = D(a,x)$, by the definition of D; so $D(a,x) < f(r)$, showing that $x \in B_D(a,f(r))$. This holds for all $x \in B_d(a,r)$; so $B_d(a,r) \subseteq B_D(a,f(r))$. On the other hand, suppose that $x \notin B_d(a,r)$. Then $d(a,x) \geq r$, and, as f is strictly increasing, $D(a,x) = f(d(a,x)) \geq f(r)$, showing that $x \notin B_D(a,f(r))$. So $x \in B_d(a,r)$ if and only if $x \in B_D(a,f(r))$. So $B_d(a,r) = B_D(a,f(r))$, as required. Since $B = B_d(a,r)$ was an arbitrary open ball in (X,d) we have shown that every open ball of (X,d) is an open ball of (X,D).

Now let B' be an arbitrary open ball in (X,D) such that $B' \neq X$. We have $B' = B_D(a,t)$ for some $t > 0$. If $t \geq 1$ then for all $x \in X$ we have that $D(a,x) < 1 \leq t$, and so $x \in B'$. This shows that B' is the whole of X, contrary to the choice of B'. So we are left with the case $t \in (0,1)$. Since f is a bijection from $[0,\infty)$ to $[0,1)$—and $f(0) = 0$—it follows that there exists $r \in (0,\infty)$ with $f(r) = t$. As shown above, in this situation $B_d(a,r) = B_D(a,f(r)) = B_D(a,t) = B'$. Since B' was an arbitrary open ball in (X,D) different from X, we have shown that every open ball in (X,D) except X is an open ball in (X,d).

4. Let X be the set of all positive integers, and for each $n \in X$ define $v(n)$ to be the largest power of 2 that is a factor of n. (Thus, for example, $v(12) = 4$ and $v(7) = 1$.) For n, $m \in X$ define

$$d(n,m) = \begin{cases} 0 & \text{if } n = m, \\ \frac{1}{v(|n-m|)} & \text{if } n \neq m. \end{cases}$$

Is d a metric on X?

Solution.

It is a metric. To show this it is sufficient (and necessary) to show that (M1), (M2) and (M3) are satisfied.

Since $|n-m| = |m-n|$ it follows that

$$d(n,m) = \frac{1}{v(|n-m|)} = \frac{1}{v(|m-n|)} = d(m,n)$$

for all m, $n \in X$ with $m \neq n$. Also $v(k) > 0$ for every positive integer k, and so $d(n,m) = 1/v(|n-m|) > 0$ for all n, $m \in X$ with $n \neq m$. If $m = n$ then obviously $d(m,n) = d(n,m)$, and $d(m,n) = 0$ by definition. So $d(m,n) = d(n,m) \geq 0$ for all m, $n \in X$. So (M1) holds.

We have just observed that $d(m,n) = 0$ if $m = n$ and $d(m,n) > 0$ if $m \neq n$; so $d(m,n) = 0$ if and only if $m = n$. That is, (M2) holds.

It remains to check (M3), the triangle inequality. Let m, n and l be arbitrary elements of X. We shall show that

$$d(l,m) + d(l,n) \geq d(m,n).$$

Note that if $l = m$ this becomes $d(m,m) + d(m,n) \geq d(m,n)$, which is trivial since $d(m,m) = 0$. Likewise if $l = n$ it becomes $d(n,m) + d(n,n) \geq d(m,n)$, which is also trivial since $d(n,n) = 0$ and $d(n,m) = d(m,n)$. Furthermore, if $m = n$ it becomes $d(l,m) + d(l,m) \geq d(m,m)$, and this is trivial too since $d(l,m) \geq 0 = d(m,m)$. So we may assume that l, m and n are all distinct.

By the unique factorization theorem for integers, every nonzero integer k can be uniquely written in the form $k = k_1k_2$, where k_1 and k_2 are integers with k_1 a power of 2 and k_2 odd. (Here k_2 is positive if and only if k is.) The number k_1 is then the largest power of 2 that is a factor of k. So, write $m - l = k_1k_2$ and $l - n = h_1h_2$, where k_2, h_2 are odd and $k_1 = 2^a$ and $h_1 = 2^b$ are powers of 2. Then $|l-m| = k_1|k_2|$ and $|l-n| = h_1|h_2|$, and so $v(|l-m|) = k_1$ and $v(|l-n|) = h_1$. Thus

$$d(l,m) = \frac{1}{k_1} + \frac{1}{h_1} = \frac{1}{2^a} + \frac{1}{2^b} \geq \frac{1}{2^c} \qquad (*)$$

where $c = \min\{a,b\}$. Now we have

$$m - n = (m-l) + (l-n) = 2^ak_2 + 2^bh_2 = 2^c(2^{a-c}k_2 + 2^{b-c}h_2),$$

and $(2^{a-c}k_2 + 2^{b-c}h_2)$ is an integer since $a \geq c$ and $b \geq c$. It follows that the largest power of 2 that is a factor of $|m-n|$ is greater than or equal to 2^c. Thus

$$d(m,n) = \frac{1}{v(|m-n|)} \leq \frac{1}{2^c} \leq d(l,m) + d(l,n)$$

by $(*)$, as required.