THE UNIVERSITY OF SYDNEY
PURE MATHEMATICS 3901

Metric Spaces 2000

Assignment 2

1. Let (X,d) and (Y,d') be metric spaces, and $f\colon X \to Y$ a function. Prove that f is continuous at the point $a \in X$ if and only if for all sequences $(x_n)_{n=0}^{\infty}$ in X, if $\lim_{n\to\infty} x_n = a$ then $\lim_{n\to\infty} f(x_n) = f(a)$.

Solution.

Suppose that f is continuous at a. Let $\varepsilon > 0$ be arbitrary. By continuity of f at a, there exists a $\delta > 0$ such that $d'(f(x), f(a)) < \varepsilon$ whenever $d(x,a) < \delta$. Since $x_n \to a$ and $n \to \infty$, there exists a positive integer N such that $d(x_n, a) < \delta$ whenever $n > N$. Now whenever $n > N$ we have $d(x_n, a) < \delta$, and hence $d'(f(x_n), f(a)) < \varepsilon$. Since ε was arbitrary, we have shown, as required, that for all $\varepsilon > 0$ there exists an N such that $d'(f(x_n), f(a)) < \varepsilon$ whenever $n > N$.

Conversely, suppose that f is not continuous at a. Then we may choose an $\varepsilon > 0$ such that for all $\delta > 0$ there exists $x \in X$ with $d(x,a) < \delta$ and $d'(f(x), f(a)) \geq \varepsilon$. Applying this with $\delta = 1/n$, we conclude that for each positive integer n we may choose $x_n \in X$ with $d(x_n, a) < 1/n$ and $d'(f(x_n), f(a)) \geq \varepsilon$. Since $0 \leq d(x_n, a) < 1/n \to 0$ as $n \to \infty$, we have that $x_n \to a$ as $n \to \infty$; however, it is not true that $f(x_n) \to f(a)$ as $n \to \infty$, since there is no value of n such that $d'(f(x_n), f(a)) < \varepsilon$.

2. Let X and Y be topological spaces and $f\colon X \to Y$ a function. Show that the following two conditions are equivalent:

(a) For all $A \subseteq X$, if A is open in X then $f(A)$ is open in Y.
(b) For all $A \subseteq X$ the inclusion $f(\operatorname{Int} A) \subseteq \operatorname{Int}(f(A))$ holds.

Solution.

Suppose that (a) holds, and let $A \subseteq X$ be arbitrary. Then $\operatorname{Int} A$ is an open subset of X, and so by (a) it follows that $f(\operatorname{Int} A)$ is open in Y. Furthermore, $\operatorname{Int} A \subseteq A$, and so $f(\operatorname{Int} A) \subseteq f(A)$. Thus $f(\operatorname{Int} A)$ is an open set contained in the subset $f(A)$ of Y. Hence

$$f(\operatorname{Int} A) \subseteq \bigcup \{U \mid U \text{ is open and } U \subseteq f(A)\} = \operatorname{Int}(f(A)),$$

and since A was an arbitrary subset of X this shows that (b) holds.

Conversely, suppose that (b) holds, and let A be an arbitrary open subset of X. Then $A = \operatorname{Int} A$, and so by (b),

$$f(A) = f(\operatorname{Int} A) \subseteq \operatorname{Int} f(A).$$

The reverse inclusion, $\operatorname{Int} f(A) \subseteq A$, is immediate from the definition of the interior of a set. So $f(A) = \operatorname{Int} f(A)$, and so $f(A)$ is open. Since A was an arbitrary open subset of X this shows that (a) holds.

3. Show that the function $\cos\colon \mathbb{R} \to \mathbb{R}$ is not a contraction mapping, but its two-fold composite $\cos^{(2)}$ is. (The metric is understood to be the usual metric on $\mathbb{R}$.) Use a calculator to find a solution of $x = \cos x$ correct to 4 decimals. (No proof required for this last bit, and not many marks awarded either!)

Solution.

Suppose that cos is a contraction mapping. Then there is a $K < 1$ such that $|\cos x - \cos y| \leq K|x - y|$ for all x, $y \in \mathbb{R}$, and so

$$\left| \frac{\cos x - \cos y}{x - y} \right| \leq K < 1$$

whenever $x \neq y$. But if we keep y fixed and let x approach y then the ratio $(\cos x - \cos y)/(x - y)$ approaches $-\sin y$, the derivative of cos at the point y. So the above inequality gives $|\sin y| \leq K < 1$, which is false for some values of y. So cos is not a contraction mapping.

Since $\frac{d}{dx}(\cos(\cos x)) = (\sin(\cos x))\sin x$, the Mean Value Theorem tells us that for all a, $b \in \mathbb{R}$ there is a $c \in [a,b]$ (or $[b,a]$ if $b < a$) such that

$$\cos^{(2)} a - \cos^{(2)} b = (a - b)(\sin(\cos c))\sin c.$$

Now $|\cos c| \leq 1$, and since sin is increasing on the interval $[-1, 1]$ (since $1 < \pi/2$) we deduce that $|\sin(\cos c)| \leq \sin 1$, and so

$$|\sin(\cos c)\sin c| \leq (\sin 1)|\sin c| \leq \sin 1,$$

irrespective of the values of a and b. So for all a, $b \in \mathbb{R}$,

$$|\cos^{(2)} a - \cos^{(2)} b| \leq (\sin 1)|a - b|,$$

which shows that $\cos^{(2)}$ is a contraction mapping, since $\sin 1 < 1$.

Since $\cos x$ takes the value 1 at $x = 0$ and 0 at $x = \pi/2$, it seems that the graphs of $y = x$ and $y = \cos x$ must cross reasonably near to $x = 0.7$. Putting $x_0 = 0.7$ and $x_i = \cos(x_{i-1})$ for all positive integers i, we find after a few iterations that 0.7391 is a good approximation to the fixed point.

4. Find metric spaces (X, d_X) and (Y, d_Y) and a function $f\colon X \to Y$ such that f is uniformly continuous and bijective, (X, d_X) is complete and (Y, d_Y) is not complete. (Modify an example from one of the the tutorial sheets.)

Solution.

Let $X = \mathbb{R}$ and $Y = (-\pi/2, \pi/2)$, a subspace of $\mathbb{R}$. (The metrics d_X and d_Y are the usual ones.) The function arctan is a uniformly continuous bijection from $\mathbb{R}$ to $(-\pi/2, \pi/2)$. Indeed, since the derivative of $\arctan x$ is $1/(1+x^2)$, the Mean Value Theorem tells us that for all x, $y \in \mathbb{R}$ there is a $c \in \mathbb{R}$ such that

$$\arctan x - \arctan y = (x-y)(1+c^2)^{-1},$$

and it follows that for all $\varepsilon > 0$ if $|x-y| < \varepsilon$ then $|\arctan x - \arctan y| < \varepsilon$. (So the definition of uniform continuity holds with δ chosen to equal ε.) Since arctan is strictly increasing it is injective, and since it is continuous and approaches $\pi/2$ as $x \to \infty$ and $-\pi/2$ as $x \to -\infty$, it maps $\mathbb{R}$ to $(-\pi/2, \pi/2)$ surjectively. We know from lectures that $X = \mathbb{R}$ is complete, whereas Y is not, since $(-\pi/2, \pi/2)$ is not closed as a subset of $\mathbb{R}$.

5. Let (X, d) be a complete metric space and $f\colon X \to X$ a function. Suppose that for some positive integer r the r-fold composite function $f^{(r)}$ (defined by $f^{(r)}(x) = f(f(f(\ldots f(x)\ldots)))$, where there are r f's on the right-hand side) is a contraction mapping. Let x be any point of X, and let $(x_n)_{n=1}^{\infty}$ be the sequence defined by $x_0 = x$ and $x_i = f(x_{i-1})$ for all positive integers i. Prove that $(x_n)_{n=1}^{\infty}$ converges in X. (You may use the fact, proved in lectures, that this is true in the case $r = 1$, or use the $r = 1$ proof as a guide to the construction of a general proof.)

Solution.

There exists a positive number $K < 1$ such that $d(f^{(r)}(x), f^{(r)}(y)) \le K d(x, y)$ for all x, $y \in X$. Since $f^{(r)}(x_i) = x_{r+i}$ (for each nonnegative integer i) it follows that

$$d(x_{nr+i}, x_{(n+1)r+i}) = d(f^{(r)}(x_{(n-1)r+i}), f^{(r)}(x_{nr+i})) \le K d(x_{(n-1)r+i}, x_{nr+i}),$$

and iterating this yields

$$\begin{aligned} d(x_{nr+i}, x_{(n+1)r+i}) &\le K d(x_{(n-1)r+i}, x_{nr+i}) \\ &\le K^2 d(x_{(n-2)r+i}, x_{(n-1)r+i}) \\ &\;\vdots \\ &\le K^n d(x_i, x_{r+i}), \end{aligned}$$

where n is any positive integer. Now if $s, t \in \mathbb{Z}^+$ with $s < t$, and if $i \in \{0, 1, \ldots, r-1\}$, then, by the triangle inequality,

$$\begin{aligned} d(x_{sr+i}, x_{tr+i}) &\le \sum_{j=0}^{t-s-1} d(x_{(s+j)r+i}, x_{(s+j+1)r+i}) \\ &\le \sum_{j=1}^{t-s} K^{s+j} d(x_i, x_{r+i}) \\ &= \frac{K^s}{1-K} d(x_i, x_{r+i}) \le \frac{K^s M}{1-K}, \end{aligned}$$

where $M = \max\{d(x_0, x_r), d(x_1, x_{r+1}), d(x_2, x_{r+2}), \ldots, d(x_{r-1}, x_{2r-1})\}$. We also have, for all nonnegative integers p, q,

$$d(x_{r+p}, x_{r+q}) = d(f^{(r)}(x_p), f^{(r)}(x_q)) \le K d(x_p, x_q),$$

and so it follows that for all i, $j \in \{0, 1, \ldots, r-1\}$ and all positive integers s,

$$\begin{aligned} d(x_{sr+i}, x_{sr+j}) &\le K d(x_{(s-1)r+i}, x_{(s-1)r+j}) \le \cdots \\ &\cdots \le K^{s-1} d(x_{r+i}, x_{r+j}) \le K^s d(x_i, x_j) \le K^s P \end{aligned}$$

where $P = \max\{\, d(x_i, x_j) \mid i, j \in \{0, 1, \ldots, r-1\}\,\}$.

Given $\varepsilon > 0$, choose s large enough so that $K^s M/(1-K)$ and $K^s P$ are both less than $\varepsilon/3$, and put $N = sr$. Let n, $m > N$ be arbitrary. Let i, $j \in \{0, 1, \ldots, r-1\}$ be the remainders obtained on dividing m, n by r, so that $m = tr + i$ and $n = ur + j$ for some integers t, $u \ge s$. Then

$$\begin{aligned} d(x_m, x_n) &\le d(x_{tr+i}, x_{sr+i}) + d(x_{sr+i}, x_{sr+j}) + d(x_{sr+j}, x_{ur+j}) \\ &\le \frac{K^s M}{1-K} + K^s P + \frac{K^s M}{1-K} \\ &\le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

Hence $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence, and hence convergent since X is complete.

Alternatively, since $f^{(r)}$ is a contraction mapping, the proof given in lectures shows that, for each $i \in \{0, 1, \ldots, r-1\}$, the sequence $(x_{nr+i})_{n=1}^{\infty}$ converges in X, the limit x being the unique fixed point of the function $f^{(r)}$. So, given $\varepsilon > 0$, there exists an integer n_i such that $d(x_{nr+i}, x) < \varepsilon$ for all $n > n_i$. Now put $N = \max\{\, n_i r + i \mid 0 \le i < r\,\}$. Let n be any integer greater than N. Choosing $i \in \{0, 1, \ldots, r-1\}$ such that $n - i$ is a multiple of r, we have $n = mr + i$ for some m, and $m > n_i$ since $n > N$. So $d(x_n, x) = d(x_{mr+i}, x) < \varepsilon$. Hence $\lim_{n\to\infty} x_n = x$.