Let $(X,\mathcal{U})$ be a topological space. Recall from Lecture 5 that if A_1 and A_2 are subsets of X such that A_2 is the complement in X of A_2, then the closure of A_2 is the complement of the interior of A_1, and the interior of A_2 is the complement of the closure of A_1. If $A = A_1$ then $A_2 = X\backslash A$; so this last statement becomes $\mathrm{Int}(X\backslash A) = X\backslash\overline{A}$.

Proposition. *Let $A \subseteq X$. Then $\overline{A} = \{\, x \in X \mid U \cap A \neq \emptyset$ for all $U \in \mathcal{U}$ with $x \in U \,\}$.*

Proof. The complement in X of $\{\, x \in X \mid U \cap A \neq \emptyset$ for all open sets U containing $x \,\}$ is the set $\{\, x \in X \mid U \cap A = \emptyset$ for some open set U containing $x \,\}$. But since the condition $U \cap A = \emptyset$ is the same as $U \subseteq (X \setminus A)$, this is just the interior of $X \setminus A$, which (as noted above) is the complement of $\overline{A}$. □

We say that a point $x \in X$ is an *accumulation point* of the subset A of X if $(U \setminus \{x\}) \cap A \neq \emptyset$ for every open neighbourhood U of x. The set of all accumulation points of A is called the *derived set* of A, and it is commonly denoted by A'. That is,

$$A' = \{\, x \in X \mid (U\backslash\{x\}) \cap A \neq \emptyset \text{ for all open neighbourhoods } U \text{ of } x \,\}.$$

We can think of accumulation points of A as those points x which have elements of $A\backslash\{x\}$ arbitrarily close by. Note that an accumulation point of A does not have to itself be a member of A: it could be in A or in $X\backslash A$.

Lemma. *Let $A \subseteq X$. Then $A' \subseteq \overline{A}$.*

Proof. Let $x \in A'$ be arbitrary. Let U be an open neighbourhood of x. Then, by the definition of A', we have $(U\backslash\{x\}) \cap A \neq \emptyset$; so $U \cap A \neq \emptyset$ (since $U\backslash\{x\} \subseteq U$). This holds for all open neighbourhoods U of x; so $x \in \overline{A}$ (by the proposition above). Since x was an arbitrary point of A', it follows that $A' \subseteq \overline{A}$, as claimed. □

Lemma. *If $x \in \overline{A}$ and $x \notin A$ then $x \in A'$.*

Proof. Suppose that $x \in \overline{A}$ and $x \notin A$, and let U be an open set with $x \in U$. By the proposition $U \cap A \neq \emptyset$, and so we may choose an element $a \in U \cap A$. Then $a \neq x$, since $x \notin A$. So $a \in U \setminus \{x\}$, as well as $a \in A$, and so we have shown that for all open neighbourhoods U of x the intersection $(U \setminus \{x\}) \cap A$ is nonempty. That is, x is an accumulation point of A. □

Proposition. *Let $A \subseteq X$. Then $\overline{A} = A \cup A'$.*

Proof. We showed above that $\overline{A} \setminus A \subseteq A'$; so $\overline{A} \subseteq A \cup A'$. For the reverse inclusion, note first that $A \subseteq \overline{A}$, by the definition of $\overline{A}$, while $A' \subseteq \overline{A}$ by one of our lemmas above. Hence $A \cup A' \subseteq \overline{A}$, as required. □

If $A \subseteq X$, we define the *frontier* or *boundary* of A to be the set $\mathrm{Fr}(A) = \overline{A} \setminus \mathrm{Int}(A)$. That is, $\mathrm{Fr}(A) = \overline{A} \cap (X\backslash\mathrm{Int}(A))$. We noted above that if B is the complement of A then $\overline{B}$ is the complement of $\mathrm{Int}(A)$; that is, $\overline{X \setminus A} = X\backslash\mathrm{Int}(A)$. So we obtain $\mathrm{Fr}(A) = \overline{A} \cap \overline{X \setminus A}$. The symmetry of this expression shows that the frontier of A is the same as the frontier of $X \setminus A$. We intuitively think of the frontier as being the set of points that are "on the edge" between A and its complement.

Proposition. *If $A \subseteq X$ then the frontier of A consists of those points $x \in X$ such that every open neighbourhood U of x has nonempty intersection with A and nonempty intersection with $X \setminus A$.*

Proof. Since $\overline{A}$ consists of the points x such that every open U containing x intersects A nontrivially, and $\overline{X \setminus A}$ consists of those x such that every open U containing x intersects $X \setminus A$ nontrivially, $\mathrm{Fr}(A)$, being the intersection of $\overline{A}$ and $\overline{X \setminus A}$, consists of those x such that both these conditions hold. □

A subset A of X is said to be *dense* if its closure is the whole space X. Now $\overline{A} = X$ if and only if every $x \in X$ has the property that every open set containing x intersects A nontrivially. Since every nonempty open set U contains some $x \in X$, it follows that $\overline{A} = X$ if and only if every nonempty open set U has nonempty intesection with A. So we have proved the following result.

Proposition. *A subset A of X is dense if and only if $A \cap U \neq \emptyset$ for every non-empty open set U.*

Dense sets are those sets which intersect all nontrivial open sets nontrivially.

A subset A of X is said to be *nowhere dense* if its closure has empty interior. That is, A is nowhere dense if and only if $\mathrm{Int}(\overline{A}) = \emptyset$.

For example, consider $\mathbb{R}$ as a topological space, the topology being determined by the usual metric on $\mathbb{R}$. If $A = \{\, 1/n \mid n \in \mathbb{Z}^+ \,\}$ then it is relatively easy to see that 0 is the only accumulation point of A, and hence $\overline{A} = A \cup \{0\}$. This set contains no open intervals, hence has no interior points. So A is nowhere dense.

Recall that a metric space is a set X together with a distance function d on X. That is, $d\colon X \times X \to \mathbb{R}$ satisfies $d(x, y) = d(y, x) \geq 0$, with $d(x, y) = 0$ if and only if $x = y$, and $d(y, z) \leq d(x, y) + d(x, z)$ for all x, y, $z \in X$. A topological space is a set X together with a collection $\mathcal{U}$ of subsets of X, such that $\emptyset$, $X \in \mathcal{U}$ and $\mathcal{U}$ is closed under finite intersections and arbitrary unions. The elements of $\mathcal{U}$ are called the open sets of the topology.

Let (X, d) be a metric space. For each $x \in X$ and each real number $\varepsilon > 0$ the open ball with centre x and radius ε is defined to be the set $B_d(x, \varepsilon)$ consisting of all points $y \in X$ such that $d(x, y) < \varepsilon$. Now define $\mathcal{U}$ to be the collection of all subsets U of X that have the following property:

$$\text{for all } x \in U \text{ there exists } \varepsilon > 0 \text{ such that } B_d(x, \varepsilon) \subseteq U. \tag{$*$}$$

As we have seen, $(X, \mathcal{U})$ is then a topological space. So every metric space is a topological space. Conversely, a topological space $(X, \mathcal{U})$ is said to be *metrizable* if it is possible to define a distance function d on X in such a way that $U \in \mathcal{U}$ if and only if the property $(*)$ above is satisfied. Note that if a topological space with at least two elements is metrizable, the metric d is not unique; for example, in Question 7 of Tutorial 2 we saw how a second metric d' can be defined in terms of a given metric d. Even more trivially, one could define $d'(x, y) = 2d(x, y)$. Two metrics on the same set X are said to be *topologically equivalent* if they give rise to the same collection of open sets $\mathcal{U}$.

There are topological spaces that are not metrizable: $\mathbb{C}^2$ with the Zariski topology, for example. Thus the concept of a topological space is weaker than that of a metric space, in that all metric spaces are topological spaces but not vice versa. Although our primary interest in this course is in metric spaces, for many of the results it suffices to assume that we are dealing with a topological space. Where possible we shall phrase proofs in such a way that they apply in the more general context.

For every positive integer n and real number $p \geq 1$ we have defined a metric d_p on the set $\mathbb{C}^n$ of all n-tuples of complex numbers. Since $\mathbb{R}^n$ is a subset of $\mathbb{C}^n$, the same definition gives a metric on $\mathbb{R}^n$. For example, when $n = 2$ the function $d_p\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

given by

$$d_p((x,y),(x',y')) = \sqrt[p]{|x-x'|^p + |y-y'|^p}$$

is a metric on $\mathbb{R}^2$. Note that in the case $p = 2$ this is the usual formula for the distance between two points in the plane. If we let p approach ∞ then $d_p((x,y),(x',y'))$ approaches $d_\infty((x,y),(x',y'))$, where the function d_∞ is defined by

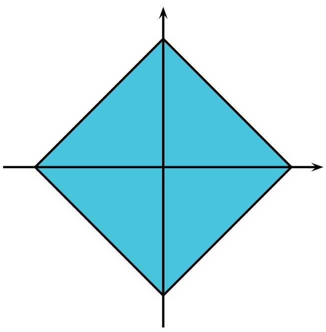
$$d_\infty((x,y),(x',y')) = \max(|x-x'|, |y-y'|).$$

This is also a metric.

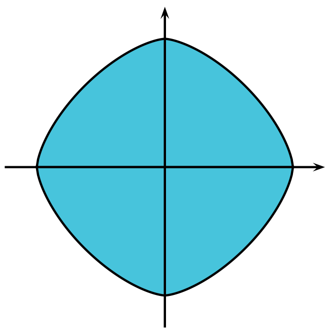
Let $a = (0,0) \in \mathbb{R}^2$. The open ball centred at a with radius 1, for the metric d_p, is the set

$$\begin{aligned} B_{d_p}(a,1) &= \{\, (x,y) \in \mathbb{R}^2 \mid d_p((x,y),(0,0)) < 1 \,\} \\ &= \{\, (x,y) \in \mathbb{R}^2 \mid \sqrt[p]{|x|^p + |y|^p} < 1 \,\} \\ &= \{\, (x,y) \in \mathbb{R}^2 \mid |x|^p + |y|^p < 1 \,\}. \end{aligned}$$

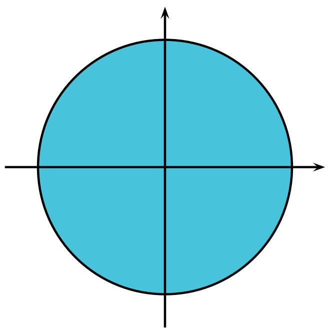
The diagram below shows this region is for various values of p.



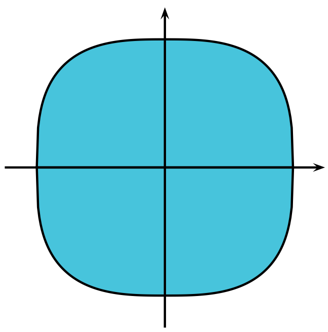
$\{\, (x,y) \bigm| |x| + |y| < 1 \,\}$
(the case p=1)



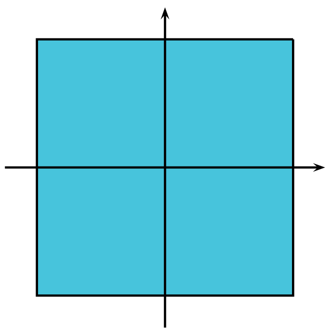
$\{\, (x,y) \bigm| |x|^{3/2} + |y|^{3/2} < 1 \,\}$
(the case p=3/2)



$\{\, (x,y) \bigm| |x|^2 + |y|^2 < 1 \,\}$
(the case p=2)



$\{\, (x,y) \bigm| |x|^3 + |y|^3 < 1 \,\}$
(the case p=3)



$\{\, (x,y) \bigm| \max(|x|, |y|) < 1 \,\}$
(the case $p=\infty$)