Some examples of topological spaces

(1) We have seen in Lectures 4 and 5 that if (X, d) is a metric space and $\mathcal{U}$ is the set of all open sets of X, where an open set (as defined in Lecture 1) is a set U with the property that for all $x \in U$ there is a $\varepsilon > 0$ with $B_d(x, \varepsilon) \subseteq U$, then $(X, \mathcal{U})$ is a topological space. The most important topological spaces are those that are derived from metric spaces in this way.

(2) Let X be a set with exactly 5 elements, a, b, c, d and e. Define

$$\mathcal{U} = \{\emptyset, \{c\}, \{b, d\}, \{a, b, d\}, \{b, d, e\}, \{b, c, d\}, \{a, b, c, d\}, \{a, b, d, e\}, \{b, c, d, e\}, X\}.$$

It is routine, if tedious, to check that the intersection of any two elements of $\mathcal{U}$ is an element of $\mathcal{U}$, and the union of any two elements of $\mathcal{U}$ is an element of $\mathcal{U}$. Using this it is easy to prove by induction that the intersection of any finite collection of elements of $\mathcal{U}$ is in $\mathcal{U}$ and the union of any finite collection of elements of $\mathcal{U}$ is in $\mathcal{U}$. Since $\mathcal{U}$ is finite an arbitrary collection of elements of $\mathcal{U}$ can only have finitely many distinct elements; so the union of any collection of elementsof $\mathcal{U}$ is in $\mathcal{U}$. Thus $\mathcal{U}$ is a topology on X (since the conditions (a), (b) and (c) of the definition given in Lecture 5 are all satisfied).

Topological spaces with only finitely many elements are not particularly important. Nevertheless it is often useful, as an aid to understanding topological concepts, to see how they apply to a finite topological space, such as X above.

(3) Let X be any infinite set, and let $\mathcal{U}$ be the set of all subsets U of X such that either $U = \emptyset$ or $X \backslash U$ is finite. That is, $\mathcal{U}$ consists of all *cofinite* subsets, along with the empty set. It is not hard to check that this is a topology. The main points are that the intersection of a finite collection of cofinite subsets of X is cofinite (because the union of a finite collection of finite sets is finite) and the union of any collection of cofinite subsets of X is cofinite (since if U is a cofinite subset of X and V any subset of X with $U \subseteq V$ then V is also cofinite).

Again, this topology is not very important, except for illustrating topological concepts.

(4) Let $X = \mathbb{C}^2$ with the Zariski topology, as defined in Lecture 5. (Recall that for this topology, a subset of X is closed if and only if it is the zero set of some collection of polynomials in two variables.) This is an example of an important topology that is not derived from a metric.

(5) Let X be any set. Let $\mathcal{A}$ be any set of subsets of X with $\emptyset \in \mathcal{A}$. Let $\mathcal{B}$ be the set of all subsets of X that are finite intersections of sets in $\mathcal{A}$. That is,

$$\mathcal{B} = \{ P_1 \cap P_2 \cap \cdots \cap P_n \mid n \in \mathbb{N} \text{ and } P_i \in \mathcal{A} \text{ for each } i \text{ from 1 to } n \}.$$

It is clear then that the intersection of any finite collection of elements of $\mathcal{B}$ is also in $\mathcal{B}$. Now put $\mathcal{U}$ equal to the set of all subsets of X that are unions of arbitrary collections of sets in $\mathcal{B}$.

It is relatively easy to check that $\mathcal{U}$ is then a topology on X. Firstly, since $\emptyset \in \mathcal{A}$ it follows that $\emptyset \in \mathcal{B}$ and hence $\emptyset \in \mathcal{U}$. By definition $\mathcal{B}$ consists of all finite intersections of sets in $\mathcal{A}$; that is, if I is a finite indexing set and $A_i \in \mathcal{A}$ for all $i \in I$ then the set

$$\bigcup_{i \in I} A_i = \{ x \in X \mid x \in A_i \text{ for all } i \in I \}$$

is in $\mathcal{B}$. It is intended that the set X is necessarily in $\mathcal{B}$. In justification of this, suppose that the indexing set I above is empty. Then the condition "$x \in A_i$ for all $i \in I$" becomes vacuous, since there are no elements i in I; so $\bigcup_{i\in I} A_i = \{\, x \in X \mid \quad \} = X$. If you do not like this justification, then you may modify the definition of $\mathcal{B}$ to include $X \in \mathcal{B}$ explicitly as an extra assumption! Given that $X \in \mathcal{B}$, it follows immediately that $X \in \mathcal{U}$.

Since any union of sets which are unions of sets in $\mathcal{B}$ is a union of sets in $\mathcal{B}$, it follows that $\mathcal{U}$ is closed under arbitrary unions. To complete the proof that $\mathcal{U}$ is a topology it only remains to show that $\mathcal{U}$ is closed under finite intersections. Furthermore, this follows easily by induction if we can prove that the intersection of two elements of $\mathcal{U}$ is in $\mathcal{U}$. Note first that if B, $B' \in \mathcal{B}$ then $B \cap B' \in \mathcal{B}$, since the intersection of two finite intersections of elements of $\mathcal{A}$ is again a finite intersection of elements of $\mathcal{A}$. Now suppose that U, U' are sets in $\mathcal{U}$. Then $U = \bigcup_{i\in I} B_i$, for some family $(B_i)_{i\in I}$ of sets in $\mathcal{B}$, and similarly $U' = \bigcup_{j\in J} B'_j$ for some family $(B'_j)_{j\in J}$ of sets in $\mathcal{B}$. This gives

$$U \cap U' = \bigcup_{\substack{i\in I\\ j\in J}} (B_i \cap B'_j),$$

which is a union of elements of $\mathcal{B}$ since $B_i \cap B'_j \in \mathcal{B}$ for all $i \in I$ and $j \in J$. That is, $U \cap U' \in \mathcal{U}$, as required.

The topology $\mathcal{U}$ that we have constructed here is the topology "generated by $\mathcal{A}$", in the sense that $\mathcal{U}$ is the smallest collection of subsets of X which is a topology and satisfies $\mathcal{A} \subseteq \mathcal{U}$. In fact Example (2) above was constructed by the method we have described in this example. Starting with

$$\mathcal{A} = \big\{\emptyset, \{c\}, \{a,b,d\}, \{b,d,e\}\big\}$$

we take all finite intersections of sets in $\mathcal{A}$ to obtain

$$\mathcal{B} = \big\{\{a,b,c,d,e\}, \emptyset, \{c\}, \{a,b,d\}, \{b,d,e\}, \{b,d\}\big\}$$

and then take all possible unions of sets in $\mathcal{B}$ to obtain the topology $\mathcal{U}$.

Separation properties

Notice that in Example (2) above, every open set U such that $b \in U$ also satisfies $d \in U$. The topology is not fine enough to distinguish between these two points. In general, if it is true in some topological space that every open set that contains the point x also contains the point y then every sequence that converges to y also converges to x. To see this, suppose that that $x_n \to y$ as $n \to \infty$, and let U be an arbitrary open set that contains x. Then U is an open set containing y, and, since $x_n \to y$ as $n \to \infty$, there exists an N such that $x_n \in U$ for all $n > N$. Since U was an arbitrary open set containing x this shows that $x_n \to x$ as $n \to \infty$, as claimed.

In almost every important topological space the above situation cannot occur: for every pair of distinct points x and y there is an open set that contains x and does not contain y. There are several similar "separation properties" that a topological space may or may not satisfy.

Here are some of the relevant definitions.

Definition. Let $(X,\mathcal{U}$ be a topological space.

(i) We say that $(X,\mathcal{U})$ is a *T_1-space* for for all x, $y \in X$ with $x \neq y$ there is an open set U with $x \in U$ and $y \notin U$.

(ii) We say that $(X,\mathcal{U})$ is a *Hausdorff space*, or a *T_2-space*, if for all x, $y \in X$ with $x \neq y$ there is an open neighbourhood U of x and an open neighbourhood V of y with $U \cap V = \emptyset$.

(iii) We say that $(X,\mathcal{U})$ is a *regular space* if whenever $x \in X$ and F is a closed set with $x \notin F$ there exist disjoint open sets U and V with $x \in U$ and $F \subseteq V$. A regular T_1-space is called a *T_3-space.*

(iv) We say that $(X,\mathcal{U})$ is a *normal space* if whenever F and G are disjoint closed sets there exist disjoint open sets U and V with $F \subseteq U$ and $G \subseteq V$. A normal T_1-space is called a *T_4-space.*

Of these, perhaps the Hausdorff condition is the most important. We may have more to say later in the course about spaces that satisfy these conditions. For the time being we content ourselves with a few observations.

Proposition. *A topological space X is a T_1-space if and only if the set $\{a\}$ is closed for all $a \in X$.*

Proof. Suppose that X is T_1, and let $a \in X$. For each $b \in X \setminus \{a\}$ the T_1 condition tells us that there is an open set neighbourhood U of b with $a \notin U$. That is, $U \subseteq X \setminus \{a\}$. So b is an interior point of $X \setminus \{a\}$. But b was an arbitrary point of $X \setminus \{a\}$; so all points of $X \setminus \{a\}$ are interior points, and so $X \setminus \{a\}$ is open.

Conversely, suppose that all singleton subsets of X are closed, and let a, $b \in X$ with $a \neq b$. Then $U = X \setminus \{b\}$ is an open set with $a \in U$ and $b \notin U$. So for every pair of distinct points of X there is an open set which contains one and not the other; that is, X is a T_1-space. □

It follows from this proposition that in a T_1 space all finite sets are closed. Indeed, $\{a_1, a_2, \ldots, a_k\} = \{a_1\} \cup \{a_2\} \cup \cdots \cup \{a_k\}$ is a finite union of closed sets, and therefore closed. Equivalently, in a T_1 space all cofinite sets are open. As we saw in Example (3) above, there is a topology (called the *cofinite topology*) such that the cofinite sets are the only nonempty open sets. This is the coarsest T_1 topology.

It is easy to see that a space which does not satisfy the T_1 condition does not satisfy the T_2 (Hausdorff) condition. We have already observed that if the T_1 condition is not satisfied, so that there are distinct points x and y such that every open set containing x also contains y, then a sequence with two distinct limits exists. In the other direction, it is not hard to show that limits are unique in Hausdorff spaces. For, suppose that X is T_2 and suppose that (x_n) is a sequence in X that converges to the point x and also to the point y. We shall show that $x = y$.

Suppose $x \neq y$. By the T_2 property we may choose open sets U and V such that $x \in U$ and $y \in V$, and $U \cap V = \emptyset$. Since $x_n \to x$ as $n \to \infty$ there exists $N \in \mathbb{Z}$ such that $x_n \in U$ for all $n > N$. Since $x_n \to y$ as $n \to \infty$ there exists $M \in \mathbb{Z}$ such that $x_n \in V$ for all $n > M$. Now if we put $n = \max\{N, M\} + 1$ then $n > N$ and $n > M$; so $x_n \in U$ and $x_n \in V$. This contradicts the fact that $U \cap V = \emptyset$. So we must have $x = y$, as desired. □

Let $\mathcal{C}$ be the set of all continuous real-valued functions on the closed interval $[0, 1]$. We shall compare various concepts of convergence for sequences in $\mathcal{C}$. Recall that d and

d', as defined below, are both metrics on $\mathcal{C}$: for all $f,\ g \in \mathcal{C}$

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|,$$
$$d'(f,g) = \int_0^1 |f(x) - g(x)|.$$

If $f \in \mathcal{C}$ and (f_n) is a sequence in $\mathcal{C}$ then to say that (f_n) converges to f in $(\mathcal{C}, d)$ is the same as saying that (f_n) converges to f uniformly on $[0,1]$. This is because the statement

$$d(f_n, f) \leq \varepsilon, \tag{1}$$

or, equivalently,

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| \leq \varepsilon,$$

is equivalent to the statement

$$|f_n(x) - f(x)| \leq \varepsilon \qquad \text{for all } x \in [0,1], \tag{2}$$

since to say that ε is an upper bound for the set $\{\, |f_n(x) - f(x)| \mid x \in [0,1] \,\}$ is equivalent to saying that ε is geater than or equal to the least upper bound of this set. So there exists an $N \in \mathbb{Z}$ such that (1) holds for all $n > N$ if and only if there exists an $N \in \mathbb{Z}$ such that (2) holds for all $n > N$.

We investigate how convergence with respect to one of these metrics relates to convergence with respect to the other. Specifically, we consider the following two questions:

(1) Does convergence in $(\mathcal{C}, d)$ imply convergence in $(\mathcal{C}, d')$?

(2) Does convergence in $(\mathcal{C}, d')$ imply convergence in $(\mathcal{C}, d)$?

We shall answer these questions next time.