Definition. A subset S of a metric space X is said to be *sequentially compact* if every infinite sequence in S has a subsequence converging to a point of S.

In Lecture 24 we proved—or claimed to prove—that a compact subset of a metric space is necessarily sequentially compact. Since there were some minor flaws in the proof given there, we start this time by presenting a suitably modified version of this proof.

Proposition. *Let C be a compact subset of a metric space (X, d). Every infinite sequence $(a_n)_{n=1}^{\infty}$ in C has a subsequence that converges to a point of C.*

Proof. Let $A = \{ a_n \mid n \in \mathbb{Z}^+ \}$, the set of points of C that occur as terms of the sequence. The sequence is infinite, but it is possible that the set A is finite; if so, there must be at least one $a \in A$ such that $a_n = a$ for infinitely many values of n. That is, in this case there exists an infinite sequence of positive integers $n_1 < n_2 < \cdots$ such that $a_{n_i} = a$ for all i. Clearly, the subsequence $(a_{n_i})_{i=1}^{\infty}$ of (a_n) is then convergent, its limit being a (which is an element of C).

We are left with the case that the set A is infinite. The proposition then guarantees the existence of a point $c \in C$ that is an accumulation point of A. Every open neighbourhood of c then contains a point of A different from c. Thus for every $\varepsilon > 0$ there exists a positive integer n such that $0 < d(c, a_n) < \varepsilon$. Let $\varepsilon_1 = 1$, and choose $n_1 \in \mathbb{Z}^+$ such that $0 < d(a_{n_1}, c) < \varepsilon_1$. (Note that in Lecture 24 we simply chose $n_1 = 1$. But it can easily be seen that for the proof to work it is necessary that $d(c, a_{n_1}) > 0$, and there is no guarantee that $a_1 \neq c$.) Now define ε_i and n_i recursively for each $i > 1$ as follows: let $\varepsilon_i = \frac{1}{2}\min\{ d(c, a_n) \mid 1 \leq n \leq n_{i-1} \text{ and } a_n \neq c \}$, and choose n_i so that $0 < d(c, a_{n_i}) < \varepsilon_i$. Since it is clear that $\varepsilon_i > 0$, such an n_i must exist. Furthermore, since $d(c, a_{n_i}) < d(c, a_n)$ for all $n \leq n_{i-1}$ such that $a_n \neq c$, it follows that either $a_{n_i} = c$ or $n_i > n_{i-1}$. But since also $d(a_{n_i}, c) > 0$ we conclude that $n_i > n_{i-1}$. Thus $(n_1, n_2, n_3, \ldots)$ is an infinite increasing sequence of positive integers, and so $(a_{n_i})_{i=1}^{\infty}$ is a subsequence of (a_n). (In Lecture 24 we essentially defined $\varepsilon_i = \frac{1}{2}d(c, a_{n_{i-1}})$; the problem with this is that it does not guarantee that $n_i > n_{i-1}$.)

A straightforward induction shows that $d(c, a_{n_i}) < 2^{-(i-1)}d(c, a_1)$ for all i, and hence $d(c, a_{n_i}) \to 0$ as $i \to \infty$. That is, the subsequence $(a_{n_i})_{i=1}^{\infty}$ of (a_n) converges to the point $c \in C$. □

The above proposition has shown that, in metric spaces, compact implies sequentially compact. We now set about proving the converse.

Let C be a sequentially compact set and let $\varepsilon > 0$. Obviously the collection of open balls of radius ε and centre in C forms an open covering of C, since each point c is in the open ball ball centered at that point. Our first lemma says that this open covering of C has a finite subcovering.

Lemma 1. *Suppose that C is a sequentially compact subset of the metric space X, and let ε be any positive number. Then there exists a finite set of points $x_1, x_2, \ldots, x_n$ of C such that $C \subseteq \bigcup_{i=1}^{n} B(x_i, \varepsilon)$.*

Proof. Suppose that no such set of points exists. We define an infinite sequence of points $x_k \in C$ recursively as follows: for each $k \in \mathbb{Z}^+$, let x_k be any point of $C \setminus \bigcup_{i=1}^{k-1} B(x_i, \varepsilon)$. The set $C \setminus \bigcup_{i=1}^{k-1} B(x_i, \varepsilon)$ is guaranteed to be nonempty (for each k) by our assumption that no finite set of points $x_i \in C$ exists with $C \subseteq \bigcup_{i=1}^{n} B(x_i, \varepsilon)$. Choosing the points in this way ensures that for all $k \in \mathbb{Z}^+$ and all $i \in \{1, 2, \ldots, k-1\}$, the point x_k is not in $B(x_i, \varepsilon)$, and therefore $d(x_i, x_k) \geq \varepsilon$.

By the assumption that C is sequentially compact, the sequence $(x_k)_{k=1}^{\infty}$ in C has a convergent subsequence. That is, there exists an infinite increasing sequence of positive integers $i_1 < i_2 < i_3 \cdots$ and a point x such that $x_{i_r} \to x$ as $r \to \infty$. It follows that there exists an integer N such that $d(x_{i_r}, x) < \varepsilon/2$ whenever $r > N$. Now let $i = i_r$ and $k = i_s$, where r, $s \in \mathbb{Z}^+$ are chosen so that $N < r < s$. Then

$$d(x_i, x_k) \le d(x_i, x) + d(x, x_k) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

contrary to the fact, explained above, that $d(x_i, x_k) \ge \varepsilon$ for all $i, k \in \mathbb{Z}^+$ with $i < k$. So our assumption was false, and the lemma is proved. □

Definition. A set of points x_1, x_2, …, $x_n \in C$ with the property that the open balls $B(x_1, \varepsilon)$, $B(x_2, \varepsilon)$, …, $B(x_n, \varepsilon)$ cover C is called an *ε-net* for C.

Lemma 1 has shown that a sequentially compact set has a ε-net, for every $\varepsilon > 0$.

Lemma 2. *Let $(U_i)_{i\in I}$ be an open covering of a sequentially compact set C. Then there exists an $\varepsilon > 0$ such that for every $x \in C$ there is an $i \in I$ for which $B(x, \varepsilon) \subseteq U_i$.*

Proof. Suppose, for a contradiction, that for every $\varepsilon > 0$ there is an $x \in C$ such that $B(x, \varepsilon)$ is not contained in any U_i. Then, in particular, for each $k \in \mathbb{Z}^+$ there exists an $x_k \in C$ such that $B(x, 1/k)$ is not contained in any U_i. Now because C is sequentially compact there is an $x \in C$ and an infinite increasing sequence of integers $k_1 < k_2 < k_3 \cdots$ such that $x_{k_n} \to x$ as $n \to \infty$. Since the U_i's cover C there is an $i \in I$ such that $x \in U_i$, and since U_i is open there exists a $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U_i$.

Since $1/k_n \to 0$ as $n \to \infty$ and $d(x_{k_n}, x) \to 0$ as $n \to \infty$, there exists an $n \in \mathbb{Z}^+$ such that $1/k_n < \varepsilon/2$ and $d(x_{k_n}, x) < \varepsilon/2$. Now, writing $k = k_n$, for all $y \in B(x_k, 1/k)$ we have

$$d(x, y) \le d(x, x_k) + d(x_k, y) < \frac{\varepsilon}{2} + \frac{1}{k} < \varepsilon,$$

and so $y \in B(x, \varepsilon) \subseteq U_i$. As this holds for all $y \in B(x_k, 1/k)$ it follows that $B(x_k, 1/k)$ is contained in U_i. This contradicts the way the x_k were chosen. □

We are now able to complete the proof of the folowing theorem.

Theorem. *Let C be a sequentially compact metric space. Then C is compact.*

Proof. Assume that C is sequentially compact, and let $(U_i)_{i\in I}$ be an arbitrary open covering of C. Choose ε as guaranteed by Lemma 2: then for all $x \in C$ there exists an $i \in I$ with $B(x, \varepsilon) \subseteq U_i$. By Lemma 1 there exist $n \in \mathbb{Z}^+$ and points x_1, x_2, …, $x_n \in C$ forming an ε-net for C; thus we have

$$C \subseteq B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \cdots \cup B(x_n, \varepsilon). \tag{1}$$

By the choice of ε we know that for each $k \in \{1, 2, \ldots, n\}$ there is an $i_k \in I$ such that $B(x_k, \varepsilon) \subseteq U_{i_k}$. By Eq. (1) it follows that

$$C \subseteq U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_n}.$$

So in the arbitrarily chosen open covering $(U_i)_{i\in I}$ we have found the finite subcovering $(U_{i_k})_{k=1}^{n}$. Hence C is compact. □

To close, we prove a generalization, to the metric space context, of the result that a continuous real-valued function on a closed and bounded interval in $\mathbb{R}$ is necessarily uniformly continuous.†

Proposition. *Let X and S be metric spaces, and suppose that X is compact. Then every continuous function $f\colon X \to S$ is uniformly continuous.*

Proof. Let $\varepsilon > 0$. For each $x \in X$ there is a $\delta_x > 0$ such that the following holds: for all $y \in X$, if $d(x, y) < \delta_x$ then $d(f(x), f(y)) < \varepsilon/2$. Since each $x \in X$ is an element of the open ball $B(x, \delta_x/2)$, it follows that the family of open balls $(B(x, \delta_x/2))_{x\in X}$ is an open covering of X. Since X is compact, there is a finite subcovering, which we may write as $(B(x, \delta_x/2))_{x\in Q}$, the set Q being a finite subset of X.

Put $\delta = \min_{x\in Q}(\delta_x/2)$. Then $\delta > 0$ (as the minimum of a finite set of positive numbers is finite), and $\delta \le \delta_x/2$ for all $x \in Q$. Now let y, $z \in X$ with $d(y, z) < \delta$. Since $(B(x, \delta_x/2))_{x\in Q}$ is a covering of X, there exists an $x \in Q$ such that $y \in B(x, \delta_x/2)$. Thus $d(y, x) < \delta_x/2$, and it follows that

$$d(z, x) \le d(z, y) + d(y, x) < \delta + \frac{\delta_x}{2} < \frac{\delta_x}{2} + \frac{\delta_x}{2} = \delta_x.$$

So $d(f(z), f(x)) < \varepsilon/2$. Since also $d(y, x) < \delta_x/2 < \delta_x$ we also have $d(f(z), f(x)) < \varepsilon/2$, and therefore

$$d(f(y), f(z)) \le d(f(y), f(x)) + d(f(x), f(z)) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This holds whenever $d(y, z) < \delta$, and since δ depends only on ε it follows that f is uniformly continuous, as required. □

† This was not in fact done in the lecture; so its proof will not be considered as part of the course for examination purposes. Nevertheless, the proof provides another good example of a compactness argument.