# Perverse sheaves and the weak Lefschetz theorem

- Today: discuss weak Lefschetz and equiv. formulations, then see how a related statement characterises perverse sheaves.

- The point: To understand a var. $X \subseteq \mathbb{P}^N$ via its hyperplane sections $X_H = X \cap H$, $H \subseteq \mathbb{P}^N$ hyperplane. We work over $\mathbb{C}$.

This is a natural instinct: consider visualising a surface in $\mathbb{R}^3$ by fixing values of one coordinate, for instance.

- Let $\mathbb{P}^N \hookrightarrow \mathbb{P}^{N'}$ denote the degree-$d$ Veronese embedding,

$$(x_0 : x_1 : \cdots : x_N) \longmapsto (x_0^d : x_0^{d-1}x_1 : \cdots : x_N^d).$$

Then hyperplane sections of $X \subseteq \mathbb{P}^{N'}$ $\Longleftrightarrow$ degree-$d$ hypersurface sections of $X \subseteq \mathbb{P}^N$.

So if $N$ is allowed to vary, hyperplane sections are as good as hypersurface sections.

- Recall: (1) $\mathcal{F} \in D^b(X)$ is constructible (or $\Lambda$-constructible for $\Lambda$ a strat. of $X$) if its cohomology sheaves are.
(2) $D^b_c(X)$ and $D^b_\Lambda(X)$ denote the respective categories.
(3) Verdier duality $\mathbb{D} : D^b_c(X) \longrightarrow D^b_c(X)$ contravariant.
(4) The perverse t-structure:

$${}^pD^{\leq 0} = \{\mathcal{F} \in D^b_c(X) : \dim \operatorname{supp} \mathcal{H}^i(\mathcal{F}) \leq -i\},$$

$${}^pD^{\geq 0} = \{\mathcal{F} \in D^b_c(X) : \dim \operatorname{supp} \mathcal{H}^i(\mathbb{D}\mathcal{F}) \leq i\}.$$

Its heart (or core) ${}^pD^{\leq 0} \cap {}^pD^{\geq 0}$ is the abelian category of perverse sheaves $P = \mathcal{M}_X$.

- <u>Notation:</u> $H^*_!$ denotes compactly supported cohomology, so e.g. if $X$ is smooth, $\dim X = n$, <u>Poincaré duality</u> is

$$H^i(X,k)^* = H_!^{n-i}(X,k).$$

<u>§1 Weak Lefschetz theorem</u> Assume $X \subseteq \mathbb{P}^N$ is smooth.

- <u>Classic weak Lefschetz (WL)</u>: Let $X_H \xhookrightarrow{i} X$. Then

$$i^*: H^*(X) \longrightarrow H^*(X_H)$$

is an isomorphism in degrees $m < \cancel{\deg}\ \dim X - 1$ and injective in degree $m = \dim X - 1$.

- Let $U = X - X_H$ be a closed subvar. of $\mathbb{P}^N - H \cong \mathbb{A}^N$, so $U$ is affine. Then have LES. in cohomology w/ compact supp.

$$\cdots \to H_!^m(U) \to H^m(X) \to H^m(X_H) \to H_!^{m+1}(U) \to \cdots$$

- This shows WL is equivalent to

$$\underset{(V!)}{H_!^m(U) = 0 \ \forall m < \dim X} \quad \overset{\text{Poincaré}}{\Longleftrightarrow} \quad \underset{(V*).}{H^m(U) = 0 \ \forall m > \dim X}$$

- Two more-powerful statements: (1) <u>Andreotti–Frankel</u>: A smooth affine variety $U/\mathbb{C}$ is homotopic to a CW complex of real dimension at most $\dim U$. [AF]

  (2) <u>Artin–Grothendieck:</u> Let $\mathcal{F}$ be a constructible sheaf on an affine variety $U$. Then $H^m(U,\mathcal{F}) = 0$ for $m > \dim U$. [Exposé XIV in SGA4.]

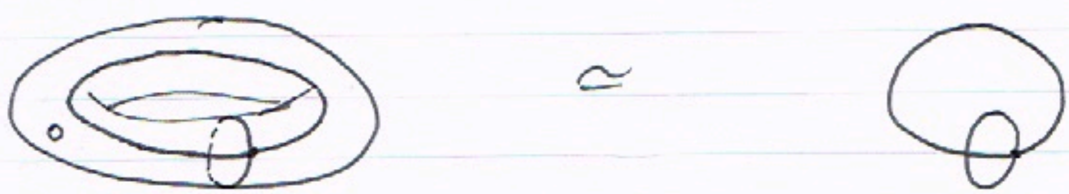
- Indeed, (1) $\Rightarrow$ ($V_*$) immediately and (2) $\Rightarrow$ ($V_*$) by taking $\mathcal{F} = \underline{\mathbb{C}}_U$ and recalling that

$$H^m(Z, \underline{\mathbb{C}}_Z) \cong H^m(Z, \mathbb{C})$$

for locally contractible ~~noncompact~~ spaces $Z$. [S]

- Example of (1): Complex projective curve $\underset{\text{homeo.}}{\cong}$ real compact surface,

so complex affine curve $\underset{\text{homotopic}}{\simeq}$ bouquet of circles.



$S^1 \times S^1 - pt.$ $\simeq$ $S^1 \vee S^1.$

- Counterexample to a variant of (2): The compactly supported variant of Artin–Grothendieck is false. If we let $i: pt \to U$ be the inclusion of a closed point and $\mathcal{F} = i_* \underline{k}$, then

$$H^0_!(U, \mathcal{F}) = k \neq 0.$$

## §2 Characterising perverse sheaves

- Prop: Let $A$ be an affine subvar. (open) in $X$. Then for $\mathcal{F} \in D^b_c(X)$ we have

$$(PV!) \quad \mathcal{F} \in {}^pD^{\geq 0} \Rightarrow H^m_!(A, \mathcal{F}) = 0 \text{ for } m < 0,$$
$$(PV*) \quad \mathcal{F} \in {}^pD^{\leq 0} \Rightarrow H^m(A, \mathcal{F}) = 0 \text{ for } m > 0.$$

- <u>Proof:</u> Consider the diagram
$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "h"'] & X \\ \operatorname{Spec}\mathbb{C} & \end{tikzcd}.$$

If $\mathcal{F} \in {}^pD^{\leq 0}(X)$ then $i^*\mathcal{F} \in {}^pD^{\leq 0}(A)$. (Easily checked.) Then

$$Rh_* i^*\mathcal{F} \in {}^pD^{\leq 0}(\operatorname{Spec}\mathbb{C}) \qquad (\text{[BBD], Thm. 4.1.1}).$$

So $\mathcal{H}^m(Rh_* i^*\mathcal{F}) = 0$ for $m > 0$

$\xrightarrow{}$ global sections on a point

$$H^m(pt, Rh_* i^*\mathcal{F}) = H^m(A, i^*\mathcal{F}) = 0 \quad \text{for } m > 0.$$

If $\mathcal{F} \in {}^pD^{\geq 0}(X)$ then $\mathbb{D}\mathcal{F} \in {}^pD^{\leq 0}(X)$ so ([HTT], C.2.16)

$$0 = H^m(A, \mathbb{D}\mathcal{F}) \cong H_!^{-m}(A, \mathcal{F})^* \quad \text{for } m > 0.$$

- Remarkably, the above conditions characterise perverse sheaves.

- <u>Theorem:</u> Let $X$ be as above, $\mathcal{F} \in D_c^b(X)$. Then $\mathcal{F} \in \mathcal{M}_X$ if and only if

$$H_!^m(A, \mathcal{F}) = 0 \quad \text{for } m < 0 \qquad \text{and} \qquad H^m(A, \mathcal{F}) = 0 \quad \text{for } m > 0$$

whenever $A \subseteq X$ is an open affine subvariety.

- <u>Proof:</u> [BBD, 4.1.6]

- Related, more general statement: Suppose $f : X \to Y$ is a quasi-finite affine map. Then $f_*({}^pD^{\leq 0}(X)) \subseteq {}^pD^{\leq 0}(Y)$, $f_!({}^pD^{\geq 0}(X)) \subseteq {}^pD^{\geq 0}(Y)$ (i.e. $f_*$ and $f_!$ are <u>right</u> and <u>left</u> <u>t-exact</u>.)

- To quote Migliorini: "Perverse sheaves are those sheaves for which [the Weak Lefschetz theorem] holds universally."

- <u>References</u>: [AF] A. Andreotti, T. Frankel, "The Lefschetz theorem on hyperplane sections," Ann. of Math. (2) 69, 1959, 713–717.

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[S] Y. Sella, "Comparison of sheaf cohomology and singular cohomology", 2016. Preprint.