

Seoul ICM 2014 - Satellite Conference on Nonlinear Elliptic and Parabolic Equations and Its Applications

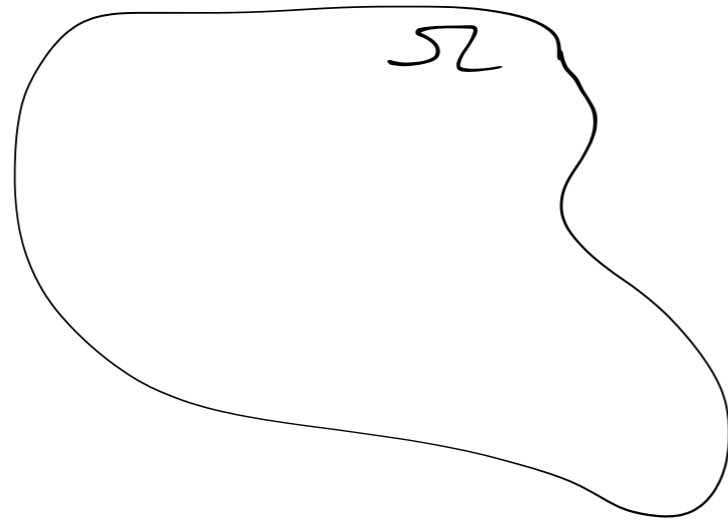
Uniform convergence of solutions to Robin & Neumann boundary problems on domains with shrinking holes

Daniel Hauer



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SYDNEY

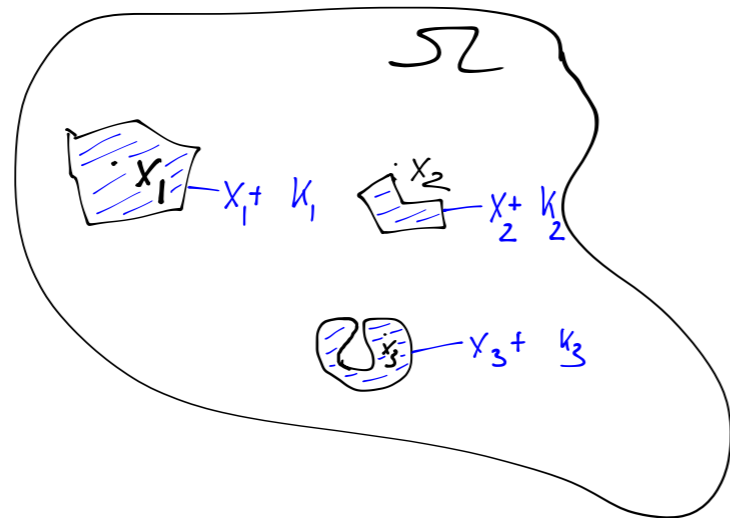
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Lipschitz domain, $d \geq 2$,



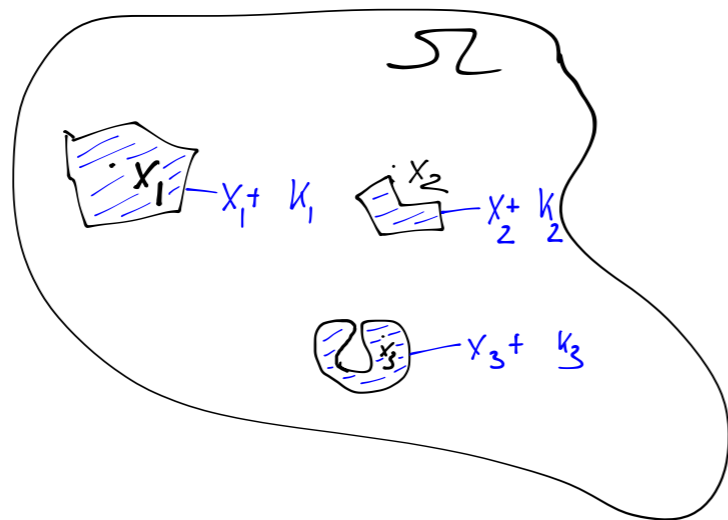
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 $\{x_1, \dots, x_m\} \subseteq \Omega$ & $K_1, \dots, K_m \subseteq \mathbb{R}^d$
nonempty compact sets with
Lipschitz bdry ∂K_j s.t.
 $\mathbb{R}^d \setminus K_j$ is connected &
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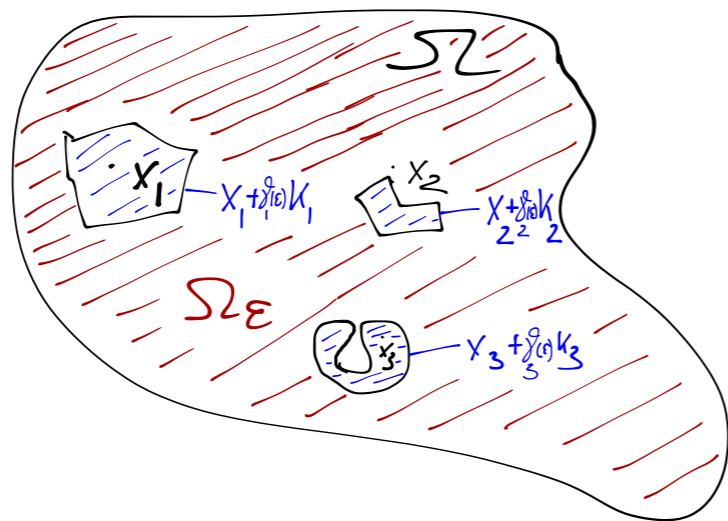
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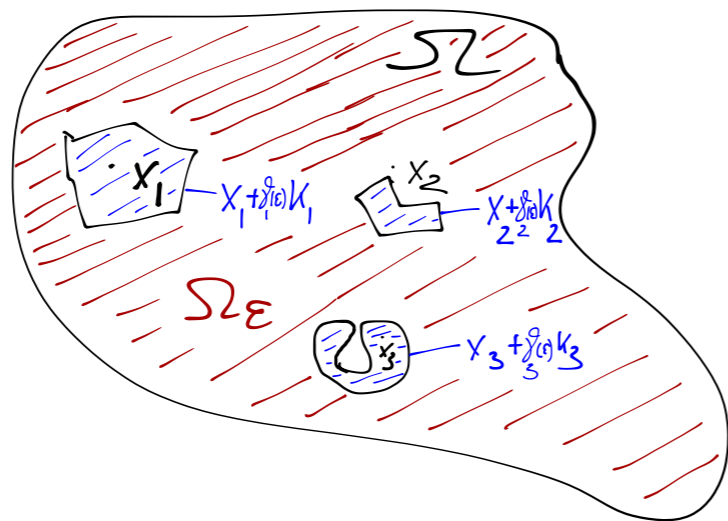
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\uparrow shrinking function
 $\delta_j: (0, 1] \rightarrow (0, 1]$
s.t. $\lim_{\varepsilon \rightarrow 0^+} \delta_j(\varepsilon) = 0$.



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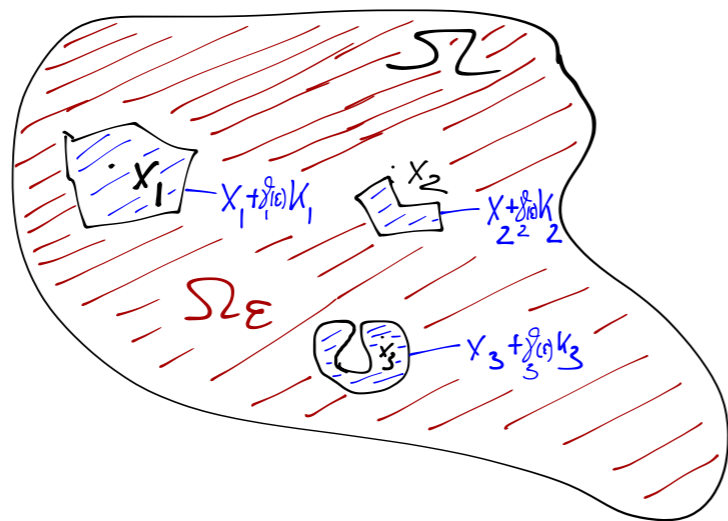


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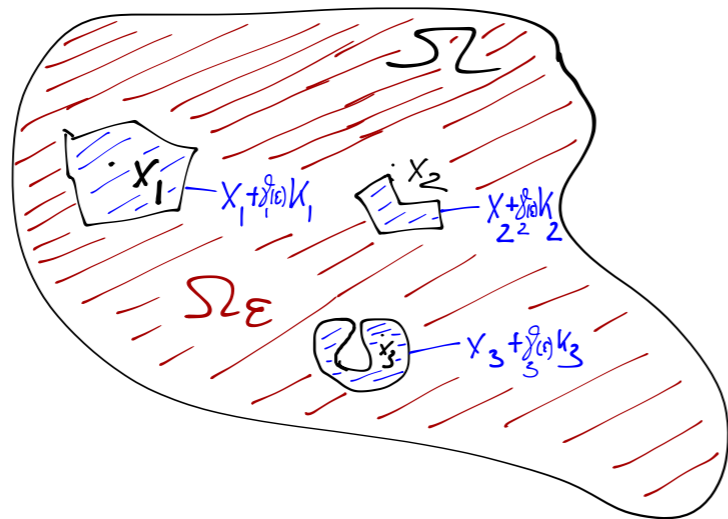
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In $d=2$, let ∂K_j be a closed curve & of class $C^{1,0}$ for $\theta \in (a, 1)$.



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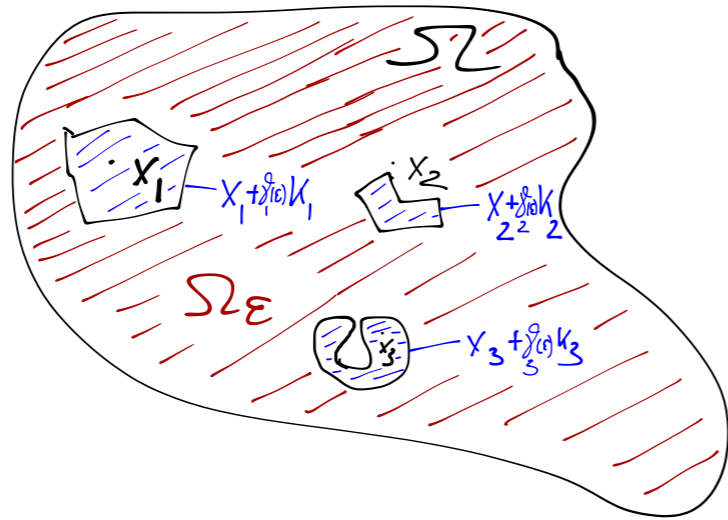


For $f_\varepsilon \in L^p(\Omega_\varepsilon)$ let u_ε be the weak solution of

$$(1) \begin{cases} -\Delta u_\varepsilon = f_\varepsilon & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \\ \frac{\partial u_\varepsilon}{\partial \nu} + \beta_\varepsilon u_\varepsilon = 0 & \text{on } \partial H_\varepsilon \end{cases}$$



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and for $f \in L^p(\Omega)$ let u be the unique weak solution of

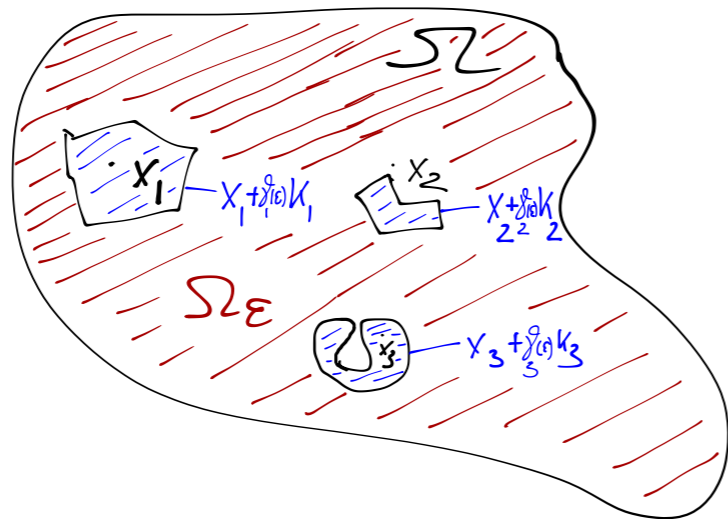
$$(2) \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Initial situation:

The aim is to prove that if $f_\varepsilon \rightarrow f$ in $L^p(\Omega)$ for some $p > \frac{d}{2}$, then the weak solutions u_ε of (1) converge uniformly to the unique weak solution u of (2)



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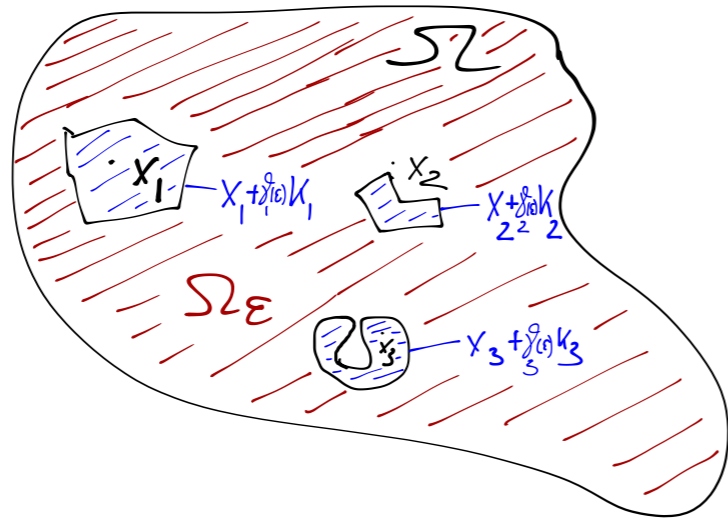
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Note: A similar result for Dirichlet problems cannot be true. ▽

Remarks

▷ L^2 - or L^p -convergence of solutions to elliptic equations on domains with shrinking holes has been studied by many authors as, for instance,

- Daners '90
- Daners-Daners '97
- Daners '99, '08
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- ▷ In contrast to this, we have established L^∞ -convergence of solutions & we do not just treat Neumann bdy conditions, but also Robin bdy conditions where β_ε may be positive or negative.



1. Theorem (Dancer, Daves, #14)

Suppose the bdy-function $\beta_\varepsilon: \partial H_\varepsilon \rightarrow \mathbb{R}$ satisfy *Assumption 1*.

If $f_\varepsilon \rightarrow f$ in $L^p(\Omega)$ for some $p > \frac{d}{2}$, then the weak solutions u_ε of

$$(1) \quad \begin{cases} -\Delta u_\varepsilon = f_\varepsilon & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \\ \frac{\partial u_\varepsilon}{\partial \nu} + \beta_\varepsilon u_\varepsilon = 0 & \text{on } \partial H_\varepsilon, \end{cases}$$

converges uniformly to the unique solution u of

$$(2) \quad \begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where "uniform convergence" means

$$\lim_{\varepsilon \rightarrow 0^+} \sup_{x \in \Omega_\varepsilon} |u_\varepsilon(x) - u(x)| = 0.$$



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For every $j=1, \dots, m$, let $\beta_\varepsilon \in L^\infty(x_j + \varepsilon \partial K_j)$ and

(1) $\exists \alpha \in (0, 1)$ s.t. $\limsup_{\varepsilon \rightarrow 0^+} \varepsilon^\alpha \|\beta_\varepsilon\|_\infty < \infty$ if β_ε is sign changing or $\beta_\varepsilon < 0$,



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Uniform estimates on domains with small holes



Uniform estimates on domains with small holes

- ▷ Solutions of elliptic bdy-value problems have a *global smoothing property* in Lebesgue spaces provided the underlying space of solutions $H^1(\Omega_\varepsilon)$ admits a "Sobolev-type inequality":

$$\|\sigma\|_{L^{d^*}(\Omega_\varepsilon)} \leq C(\Omega_\varepsilon) \|\sigma\|_{H^1(\Omega_\varepsilon)} \quad \forall \sigma \in H^1(\Omega_\varepsilon),$$

where $d^* = \frac{2d}{d-2}$ if $d \geq 3$, & $d^* < \infty$ if $d=2$ &

$$C(\Omega_\varepsilon) := \sup \left\{ \|\sigma\|_{L^{d^*}(\Omega_\varepsilon)} \mid \|\sigma\|_{W^1(\Omega_\varepsilon)} = 1 \right\}.$$



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The problem of such a $M \geq 0$ has been studied intensively by Daners [Thm. 2.8, Dan99] for domains with shrinking holes & $\beta_\varepsilon \geq 0$.



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Uniform estimates on domains with small holes

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2. Theorem (Dancer, Daners, H., 14)

Suppose that **Assumption 1** is satisfied. Let $p > \frac{d}{2}$ and $f_\varepsilon \in L^p(\Omega_\varepsilon)$ for all $\varepsilon \in (0, \varepsilon_0]$.

Then $\exists \varepsilon_0 > 0$ & $M > 0$ independent of $\varepsilon \in (0, \varepsilon_0]$

such that the solution u_ε of

$$(1) \quad \begin{cases} -\Delta u_\varepsilon = f_\varepsilon & \text{in } \Omega_\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial\Omega \\ \frac{\partial u_\varepsilon}{\partial \nu} + \beta_\varepsilon u_\varepsilon = 0 & \text{on } \partial H_\varepsilon \end{cases}$$

satisfies

$$\|u_\varepsilon\|_\infty \leq M \cdot \|f_\varepsilon\|_p \quad \text{for all } \varepsilon \in (0, \varepsilon_0]$$



Consequences of uniform L^p - L^∞ -estimates

- ▷ The convergence of u_ε to u in $L^p(\Omega)$ as $\varepsilon \rightarrow 0+$.



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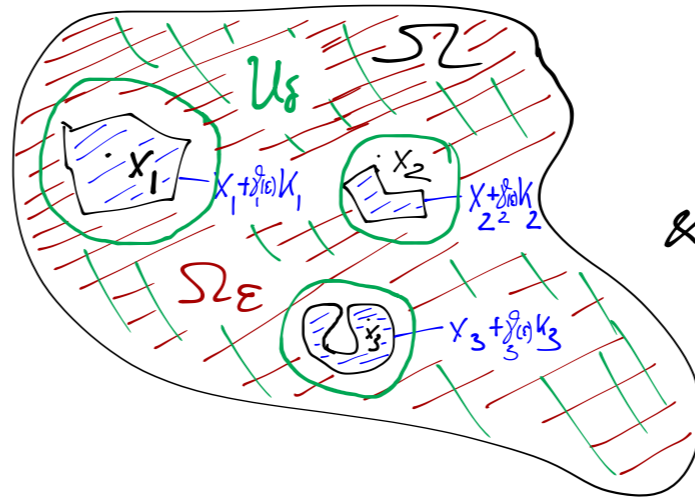


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- ▷ The convergence of u_ε to u in $L^p(\Omega)$ as $\varepsilon \rightarrow 0+$.
- ▷ Uniform convergence away from the holes;

\implies The proof of Theorem 1 reduces to showing that $\forall \delta > 0$ we have

This means that the initial problem becomes a "local" problem. $\left\{ \sup_{x \in \Omega_\varepsilon \setminus U_\delta} |u_\varepsilon(x) - u(x)| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0+ \right.$



& $\forall \delta > 0$ let

$$U_\delta := \Omega \setminus \bigcup_{j=1}^m B(x_j, \delta).$$

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- ▷ We may assume that u_ε & u are harmonic close to the holes.



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\Rightarrow We need to deal with only one K_i at a time.
By translation, we may assume that $x_i = 0$.



Consequences of uniform L^p - L^∞ -estimates

\Rightarrow We need to deal with only hole K_1 at a time.
By translation, we may assume that $x_1 = 0$.

\curvearrowright The new problem which need to consider instead of problem (1) is:

$$\tilde{(1)} \quad \begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \tilde{\Omega}_\varepsilon := B(0,1) \setminus \varepsilon K_1 \\ \frac{\partial u_\varepsilon}{\partial \nu} + \beta_\varepsilon u_\varepsilon = 0 & \text{on } \varepsilon \partial K_1 \end{cases}$$



Rescaling to a hole of fixed size



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We set $\vartheta_\varepsilon(x) := u_\varepsilon(\varepsilon \cdot x)$, $\tilde{\beta}_\varepsilon(x) := \beta_\varepsilon(\varepsilon x)$.



Rescaling to a hole of fixed size

We set $v_\varepsilon(x) := u_\varepsilon(\varepsilon \cdot x)$, $\tilde{\beta}_\varepsilon(x) := \beta_\varepsilon(\varepsilon x)$.

→ v_ε is a weak solution of

$$(3) \quad \begin{cases} -\Delta v_\varepsilon = 0 & \text{in } \varepsilon^{-1} \tilde{\Omega}_\varepsilon = B(0, \frac{1}{\varepsilon}) \setminus K_1 \\ \frac{\partial v_\varepsilon}{\partial \nu} + \varepsilon \tilde{\beta}_\varepsilon v_\varepsilon = 0 & \text{on } \partial K_1 \end{cases}$$

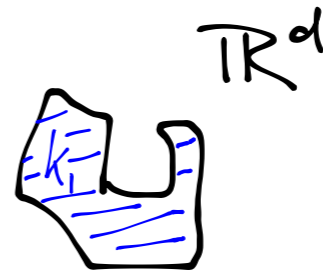
satisfying $\|v_\varepsilon\|_\infty = \|u_\varepsilon\|_\infty \leq M \quad \forall \varepsilon \in (0, \bar{\varepsilon})$.



Letting $\varepsilon \rightarrow 0+$ in the rescaled problem



$\varepsilon \rightarrow 0+$
→



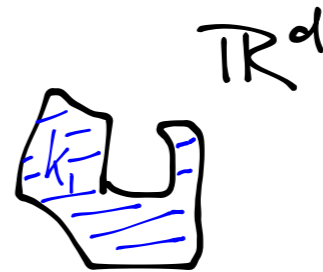
Exterior domain



Letting $\varepsilon \rightarrow 0+$ in the rescaled problem



$\varepsilon \rightarrow 0+$
 \longrightarrow



Exterior domain

Since $\|\vartheta_\varepsilon\|_\infty \leq M \forall \varepsilon > 0$ & by Assumption 1 on β_ε :

$\hookrightarrow (\vartheta_\varepsilon)$ is bdd in H^1_{loc} , $\hookrightarrow \vartheta_\varepsilon \rightarrow \vartheta$ in H^1_{loc}
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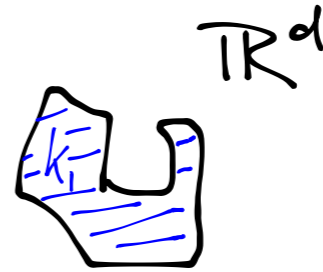
$$\text{of } \begin{cases} -\Delta \vartheta = 0 & \text{in } K_1^c \\ \frac{\partial \vartheta}{\partial \nu} = 0 & \text{on } \partial K_1 \end{cases}$$



Letting $\varepsilon \rightarrow 0+$ in the rescaled problem



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Exterior domain

Since $\|v_\varepsilon\|_{L^\infty} \leq M \forall \varepsilon > 0$ & by Assumption 1 on β_ε :

$\hookrightarrow (v_\varepsilon)$ is bdd in H^1_{loc} , $\hookrightarrow v_\varepsilon \rightarrow v$ in H^1_{loc}
 v is a weak solution

$$\text{of } \begin{cases} -\Delta v = 0 & \text{in } K_1^c \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial K_1 \end{cases}$$

Liouville's lemma ([Danos, Daners, H.'13])

$\implies v \equiv v_0$ on K_1^c for some $v_0 \in \mathbb{R}$.



To prove our Main Theorem
we need to identify \mathcal{V}_0 .



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At this point, we need to treat the case $d \geq 3$
and $d=2$ separately.



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Let $x = (r, \theta)$ where $r = |x|$ & $\theta \in \mathbb{S}_d$ being the unit sphere in \mathbb{R}^d .



Proof of Theorem 1 for $d \geq 3$:

▷ We apply a transformation in polar coordinates to $v_\varepsilon, u_\varepsilon$ & h :

Let $x = (r, \theta)$ where $r = |x|$ & $\theta \in \mathbb{S}_d$ being the unit sphere in \mathbb{R}^d .

If v is a harmonic function on an annulus

$$A_{r_1, r_2} := \{x \in \mathbb{R}^d \mid r_1 < |x| < r_2\}$$

for some $0 < r_1 < r_2$,

then $\bar{v}(r) := \int_{\mathbb{S}_d} v(r, \theta) d\theta \quad \forall r \in (r_1, r_2)$

is a solution of $\frac{d}{dr} \left(r^{d-1} \frac{dv}{dr} \right) = 0$ on (r_1, r_2) .



Proof of Theorem 1 for $d \geq 3$:

▷ We apply a transformation in polar coordinates to $v_\varepsilon, u_\varepsilon$ & h :

$$\implies \bar{v}_\varepsilon \text{ satisfies } \frac{d}{dr} \left(r^{d-1} \frac{d\bar{v}}{dr} \right) = 0 \text{ on } (R_1, \frac{1}{\varepsilon})$$

where $R_1 := \sup\{|y| \mid y \in K_1\} < 1$ ^{by assumption}

\bar{u}_ε & \bar{u} satisfy this equation on $(\varepsilon R_1, 1)$.



Proof of Theorem 1 for $d \geq 3$:

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\bar{v}_ε & \bar{u} satisfy this equation on $(\varepsilon R_1, 1)$.

Since $\{1, E_d\}$ is a basis of the space of radially symmetric harmonic functions on any annulus & $v_\varepsilon(x) = u_\varepsilon(\varepsilon x)$

$$\begin{aligned} \implies \bar{v}_\varepsilon(r) &= a_\varepsilon + b_\varepsilon E_d(r) \\ \bar{u}_\varepsilon(r) &= a_\varepsilon + b_\varepsilon E_d(r/\varepsilon) = a_\varepsilon + b_\varepsilon \varepsilon^{d-2} E_d(r) \\ \bar{u}(r) &= u(0) \text{ (Mean value property)} \end{aligned}$$



Proof of Theorem 1 for $d \geq 3$:



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Now, let $\varepsilon_2 \rightarrow 0^+$.



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& hence $\overline{v_{\varepsilon_2'}}(\tau) \rightarrow v_0 \quad \forall \tau > 1.$



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From (1) & since $\{1, E_d\}$ is linear independent

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$$\Rightarrow b_{\varepsilon_2'} \rightarrow 0 \quad \& \quad a_{\varepsilon_2'} \rightarrow v_0$$

$$\& \text{ so also } \varepsilon^{d-2} b_{\varepsilon_2'} \rightarrow 0.$$

$$\text{Using } \bar{u}_\varepsilon(\tau) = a_\varepsilon + b_\varepsilon \varepsilon^{d-2} E_d(\tau) \stackrel{(2)}{\Rightarrow} \underline{\underline{u(0) = v_0}}$$

□



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We can complete the proof if we assume that

$$\mathbb{R}^2 \setminus K_1 \cong \mathbb{R}^2 \setminus B(0,1)$$

& by using the boundary cond. on $\partial \varepsilon K_1$. \square



Thank You!

