On the Multifractal Formalism

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Conference in honour of Gavin Brown
Sydney, March 5–6, 2007

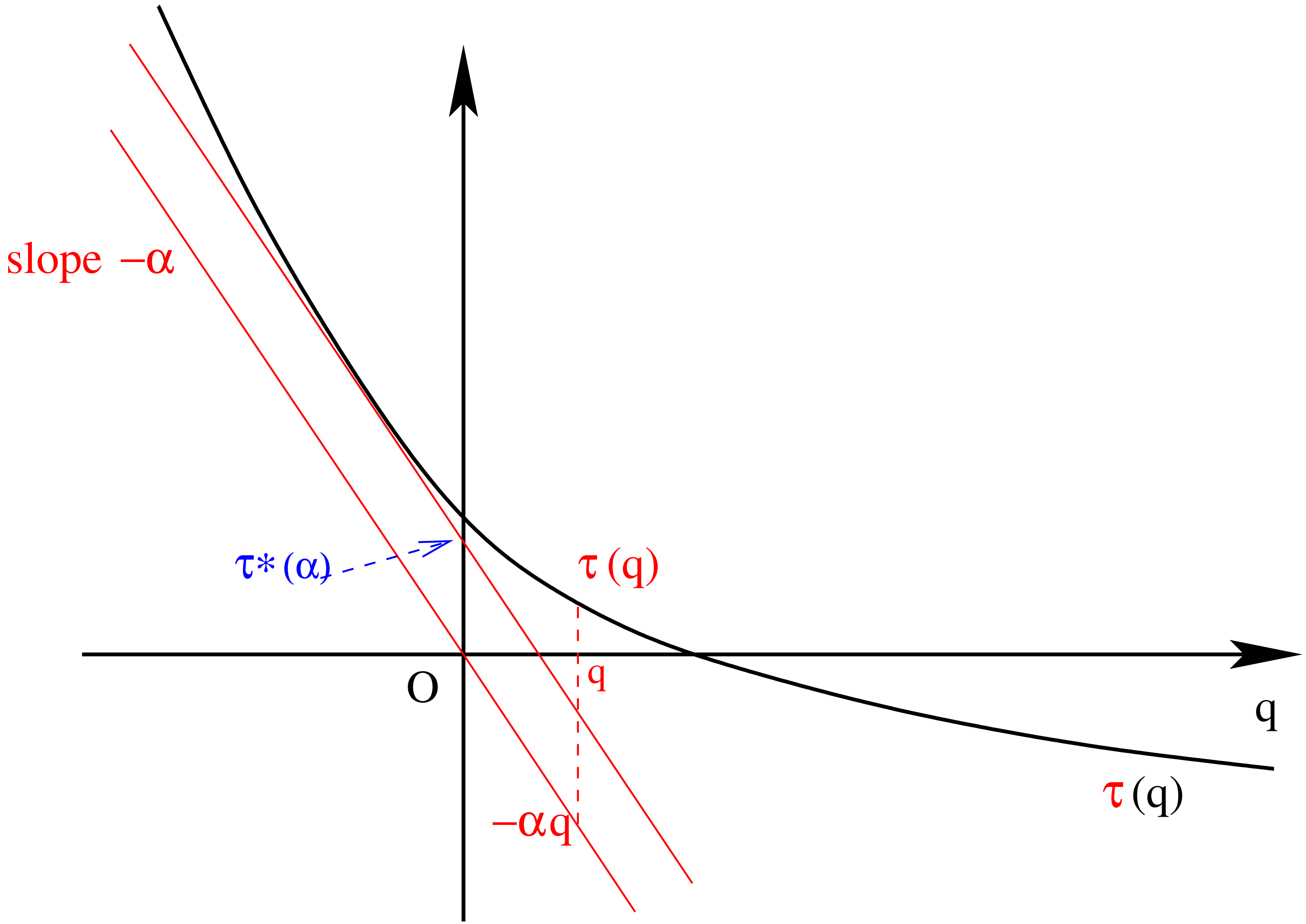
The Brown-Michon-Peyrière 1992 paper

μ : probability measure on $[0,1]$
$I_{n,j}$: the jth c-adic interval of length c^n $(0 \le j < c^n)$
$I_n(x)$: the c-adic interval of length c^n containing x

$$E_\alpha = \left\{ x \in [0,1] \; ; \; \lim_{n\to\infty} \frac{\log \mu\Big(I_n(x)\Big)}{-n \log c} = \alpha \right\}$$

$$\tau(q) = \lim_{n\to\infty} \frac{1}{n \log c} \log \sum_{j=0}^{c^n-1} \mu\Big(I_{n,j}\Big)^q$$

Then $\dim E_\alpha = \tau^*(\alpha) = \inf_{t\in\mathbb{R}} \tau(t) + \alpha\, t$
if $\alpha = -\tau'(q)$ and if there exists a measure μ_q such that
$C^{-1}\mu(I)^q c^{-n\tau(q)} \le \mu_q(I) \le C\,\mu(I)^q c^{-n\tau(q)}$ for any c-adic interval of order n



The Legendre Transform : $\tau^*(\alpha) = \inf_{q\in\mathbb{R}} \tau(q) + \alpha q$

The setting in BMP is more general:

$\Big\{\{I_{n,j}\}_{0\le j<N_n}\Big\}_{n>0}$ is a sequence of nested partitions of $[0,1)$ by semi-open intervals.

Set $C_n(q,t) = \sum_j \mu\Big(I_{n,j}\Big)^q |I_{n,j}|^t$

and $C(q,t) = \limsup_{n\to\infty} C_n(q,t)$

The boundary of the convex set $\{(q,t)\ ;\ C(q,t)=0\}$ is the graph of a function τ, which is convex and non-increasing.

Then

always	$\dim E_\alpha \le \tau^*(\alpha)$,
sometimes	$\dim E_\alpha = \tau^*(\alpha)$.

Hausdorff measures and dimension

Let $(\mathbb{X}, d)$ be a metric space.

$B(a,r) = \{x \in \mathbb{X} \mid d(a,x) \leq r\}$

For $A \subset \mathbb{X}$, $t > 0$ and $\delta > 0$

$$
\begin{aligned}
\mathcal{H}^t_\delta(A) &= \inf \left\{ \sum r_j^t \mid A \subset \bigcup B(x_j, r_j),\ r_j \leq \delta \right\} \\
\mathcal{H}^t(A) &= \lim_{\delta \searrow 0} \mathcal{H}^t_\delta(A) \\
\dim A &= \inf\{t \geq 0 \mid \mathcal{H}^t(A) = 0\} \\
&= \sup\{t \geq 0 \mid \mathcal{H}^t(A) = +\infty\}
\end{aligned}
$$

A general setting

ξ : a positive function defined on the balls of $\mathbb{R}^n$

$$X(\alpha) = \left\{ x \ ; \ \lim_{r \searrow 0} \frac{\log \xi(B(x,r))}{\log r} = \alpha \right\}$$

Task : to compute the dimension of $X(\alpha)$; more precisely, to express $\alpha \mapsto \dim X(\alpha)$, as a Legendre transform.

Common choices

- ξ is a measure,
  (this is the case considered in [BMP], with boxes instead of balls)

- $\xi\Big(B(x,r)\Big)$ is the modulus of continuity at x of a function.

Indeed, one could think of other choices, e.g.

– a Choquet capacity

– $\xi\Big(B(x,r)\Big) = \int_{B(x,r)} \left| f(y) - \frac{1}{|B(x,r)|} \int_{B(x,r)} f(z)\,\mathrm{d}z \right| \mathrm{d}y$

One could also wish to perform simultaneous analysis of several functions ξ. Expressions such as

$$\sum \xi_1(B_j)^{q_1} \xi_2(B_j)^{q_2} \cdots \xi_k(B_j)^{q_k} |B_j|^t$$

would be involved.

To be able to consider infinitely many ξ's at a time, it is better to write $\xi = \exp -\varkappa$.

Let $(\mathbb{X}, d)$ be a metric space satisfying the Besicovitch covering property.

$$B(a,r) = \{x \in \mathbb{X} \mid d(a,x) \leq r\}$$

We are given a function $\varkappa$ from $\mathbb{X} \times \mathbb{R}^+$ to $\mathbb{E}'$, the dual of a separable real Banach space $\mathbb{E}$. We denote by $\langle\ ,\ \rangle$ the duality bracket between $\mathbb{E}$ and $\mathbb{E}'$.

We are going to define several quantities and sets, as L. Olsen.

Multifractal Hausdorff measures

For $A \subset \mathbb{X}$, $q \in \mathbb{E}$, $t \in \mathbb{R}$, and $\delta > 0$, we set

$$\overline{\mathcal{H}}^{q,t}(A) = \inf \sum_j \mathrm{e}^{-\left(\langle q, \varkappa(x_j, r_j)\rangle - t \log r_j\right)},$$

where the infimum is taken over the families $\{(x_j, r_j)\}$ such that $\{B(x_j, r_j)\}$ is a centered δ-cover of A,

$$\overline{\mathcal{H}}^{q,t}(A) = \lim_{\delta \searrow 0} \overline{\mathcal{H}}(A), \text{ and } \mathcal{H}^{q,t}(A) = \sup_{F \subset A} \overline{\mathcal{H}}^{q,t}(F).$$

When $\varkappa = 0$, these measures reduce to the usual Hausdorff measures.

If $\overline{\mathcal{H}}^{q,t}(A) < \infty$, then for all $s > t$, $\overline{\mathcal{H}}^{q,s}(A) = 0$, so there is a critical index t_0 such that $\overline{\mathcal{H}}^{q,t}(A) = 0$ for $t > t_0$ and $\overline{\mathcal{H}}^{q,s}(A) = \infty$ for $t < t_0$.

Packing measures

For $A \subset \mathbb{X}$, $q \in \mathbb{E}$, $t \in \mathbb{R}$, and $\delta > 0$, we set

$$\overline{\mathcal{P}}_{\delta}^{q,t}(A) = \sup \sum_{j} \mathrm{e}^{-\left(\langle q, \varkappa(x_j, r_j)\rangle - t \log r_j\right)},$$

where this supremum is taken on collections $\{(x_j, r_j)\}$ such that $r_j \leq \delta$ and $\{B(x_j, r_j)\}$ is a centered δ-packing of A.

$$\overline{\mathcal{P}}^{q,t}(A) = \lim_{\delta \searrow 0} \overline{\mathcal{P}}_{\delta}^{q,t}(A),$$

$$\mathcal{P}^{q,t}(A) = \inf \left\{ \sum_{j} \overline{\mathcal{P}}^{q,t}(F_j) \mid A \subset \bigcup_{j} F_j \right\}.$$

One defines, as Olsen,

$$\mathsf{B}(q) = \inf\{t \in \mathbb{R} \mid \mathcal{P}^{q,t}(\mathbb{X}) = 0\},$$

and

$$\mathsf{b}(q) = \inf\{t \in \mathbb{R} \mid \mathcal{H}^{q,t}(\mathbb{X}) = 0\}.$$

We have the inequality $\mathsf{b} \leq \mathsf{B}$.

Proposition 1. *The function* B *is convex.*

Proof. Let p, $q \in \mathbb{E}$, $t > \mathsf{B}(p)$, and $u > \mathsf{B}(q)$.

So, for all $n \geq 1$, $\mathcal{P}^{p,t}(\mathbb{X}) = \mathcal{P}^{q,u}(\mathbb{X}) = 0$. One can write $\mathbb{X} = \bigcup_{j\geq 1} A_j = \bigcup_{k\geq 1} F_k$ so that $\sum_{j\geq 1} \overline{\mathcal{P}}^{p,t}(A_j) \leq 1$ and $\sum_{k\geq 1} \overline{\mathcal{P}}^{q,u}(F_k) \leq 1$. Then, for all $\alpha \in (0,1)$

$$\overline{\mathcal{P}}^{\alpha p+(1-\alpha)q,\alpha t+(1-\alpha)u}(A_j \cap F_k) \leq \left(\overline{\mathcal{P}}^{p,t}(A_j \cap F_k)\right)^{\alpha} \left(\overline{\mathcal{P}}^{q,u}(A_j \cap F_k)\right)^{1-\alpha}$$

Then, due to the Hölder inequality, one has

$$\begin{aligned}
\sum_{1\leq j,k\leq m} & \overline{\mathcal{P}}^{\alpha p+(1-\alpha)q,\alpha t+(1-\alpha)u}(A_j \cap F_k) \\
&\leq \left(\sum_{1\leq j,k\leq m} \overline{\mathcal{P}}^{p,t}(A_j \cap F_k)\right)^{\alpha} \left(\sum_{1\leq j,k\leq m} \overline{\mathcal{P}}^{q,u}(A_j \cap F_k)\right)^{1-\alpha} \\
&\qquad \leq \left(m \sum_{1\leq j\leq m} \overline{\mathcal{P}}^{p,t}(A_j)\right)^{\alpha} \left(m \sum_{1\leq k\leq m} \overline{\mathcal{P}}^{q,u}(F_k)\right)^{1-\alpha} \leq m.
\end{aligned}$$

It results that

$$\mathcal{P}^{\alpha p+(1-\alpha)q,\alpha t+(1-\alpha)u}\left(\bigcup_{1\le j,k<m} A_j \cap F_k\right) \le m.$$

Therefore, if $\varepsilon > 0$,

$$\mathcal{P}^{\alpha p+(1-\alpha)q,\alpha t+(1-\alpha)u+\varepsilon}(\mathbb{X}) = 0$$

and

$$\mathsf{B}\Big(\alpha p + (1-\alpha)q\Big) \le \alpha t + (1-\alpha)u + \varepsilon$$

Local Hölder exponent – Chernoff-like inequalities

For $\alpha \in \mathbb{E}'$ and $E \subset \mathbb{E}$, we set

$$X(\alpha, E) = \left\{ x \mid \limsup_{r \searrow 0} \frac{\langle w, \varkappa(x,r) \rangle}{-\log r} \leq \langle w, \alpha \rangle \text{ for all } w \in E \right\}.$$

$X(\alpha, \mathbb{E})$, simply denoted by $X(\alpha)$, is the set of points x such that $\lim_{r \searrow 0} \frac{\varkappa(x,r)}{-\log r} = \alpha$ (in the $\sigma(\mathbb{E}, \mathbb{E}')$ topology).

Proposition 2. $\mathrm{Dim}\, X(\alpha, \{q\}) \leq \langle q, \alpha \rangle + \mathrm{B}(q)$.

Corollary 3. *For $\alpha \in \mathbb{E}'$ and $E \subset \mathbb{E}$, one has*

$$\mathrm{Dim}\, X(\alpha, E) \leq \inf_{q \in E} \langle q, \alpha \rangle + \mathrm{B}(q).$$

$\mathrm{Dim}\, X(\alpha) \leq \inf_{q \in \mathbb{E}} \langle q, \alpha \rangle + \mathrm{B}(q) = B^*(\alpha)$ (Legendre transform).

Proof. Let $\varepsilon > 0$, $\eta > 0$, $q \in \mathbb{E}$, $m \geq 1$.

Set $\quad A_m(\varepsilon) = \left\{ x \in \mathbb{X} \mid \frac{\langle q, \varkappa(x,r)\rangle}{-\log r} \leq \langle q, \alpha\rangle + \varepsilon \quad \text{for} \quad r < 1/m \right\}.$
Let $\{B(x_j, r_j)\}$ be a δ-packing of $F \subset A_m(\varepsilon)$, with $\delta < 1/m$. One has

$$\sum_j \mathrm{e}^{\left(\langle q,\alpha\rangle + \varepsilon + \mathsf{B}(q) + \eta\right)\log r_j} \leq \sum_j \mathrm{e}^{-\left(\langle q, \varkappa(x_j, r_j)\rangle - \log(r_j)(\mathsf{B}(q)+\eta)\right)},$$

so

$$\overline{\mathcal{P}}^{\langle q,\alpha\rangle + \varepsilon + \mathsf{B}(q) + \eta}(F) \leq \overline{\mathcal{P}}^{q,\mathsf{B}(q)+\eta}(F).$$

Since $\mathcal{P}^{q,\mathsf{B}(q)+\eta}(\mathbb{X}) = 0$,
$\inf\left\{ \sum_j \overline{\mathcal{P}}^{q,\mathsf{B}(q)+\eta}(F_j) \mid \mathbb{X}_n \subset \bigcup F_j \right\} = 0$. It results

$$\mathcal{P}^{\langle q,\alpha\rangle + \varepsilon + \mathsf{B}(q) + \eta}\left(A_m(\varepsilon)\right) = 0.$$

Since $\mathcal{P}^{\langle q,\alpha\rangle+\varepsilon+\mathsf{B}(q)+\eta}\Big(A_m(\varepsilon)\Big) = 0$ for any $\eta > 0$,

$\operatorname{Dim} A_m(\varepsilon) \le \langle q,\alpha\rangle + \varepsilon + \mathsf{B}(q)$. But as

$$\left\{x \in \mathbb{X} \mid \limsup_{r\searrow 0} \frac{\langle q, \varkappa(x,r)\rangle}{-\log r} \le \langle q,\alpha\rangle\right\} \subset \bigcap_{p\ge 1} \bigcup_{m\ge 1} A_m(1/p),$$

we get the announced inequality.

Remark. If the formula gives a negative dimension, this means that the corresponding set is empty.

Proposition 4. *Set*

$$X^*(\alpha, E) = \left\{x \mid \liminf_{r\searrow 0} \frac{\langle w, \varkappa(x,r)\rangle}{-\log r} \le \langle w,\alpha\rangle \text{ for all } w \in E\right\}.$$

Then

$$\dim X^*(\alpha, E) \le \inf_{q\in E} \langle q,\alpha\rangle + \mathsf{B}(q).$$

The converse inequality

Notations:

- If $|\mathsf{B}(q)| < \infty$ and $v \in \mathbb{E}$, one sets

$$\partial_v \mathsf{B}(q) = \lim_{t \searrow 0} \frac{\mathsf{B}(q+tv) - \mathsf{B}(q)}{t};$$

- $\mathsf{B}'(q)$ stands for the derivative (considered as an element of $\mathbb{E}'$) of B at point q, when it exists.

When B has a partial derivative at point q along the direction v, one has $\partial_{-v}\mathsf{B}(q) = -\partial_v \mathsf{B}(q)$.

When $\mathsf{B}'(q)$ exists, $\partial_v \mathsf{B}(q) = \langle v, \mathsf{B}'(q) \rangle$.

Lemma 5. *Let $v \in \mathbb{E}$ and q such that $|\mathsf{B}(q)| < \infty$. Then*

$$\mathcal{H}^{q,\mathsf{B}(q)} \left\{ x \mid \liminf_{r \searrow 0} \frac{\langle v, \varkappa(x,r) \rangle}{-\log r} < -\partial_v \mathsf{B}(q) \right\} = 0.$$

Lemma 6. *Let $x \in \mathbb{X}$. Consider the function $\rho_x(v) = \liminf_{r \searrow 0} \frac{\langle v, \varkappa(x,r) \rangle}{-\log r}$ and the cone $C_x = \{v \in \mathbb{E} \mid \rho_x(v) > -\infty\}$. The function ρ_x is concave and the cone C_x is convex. If the interior C_x° of C_x is nonempty two alternatives may occur: either $\rho_x(v) = +\infty$ for one $v \in C_x^\circ$ and then $\rho_x(v) = +\infty$ for all $v \in C_x^\circ$, or ρ_x is continuous on C_x°.*

Proposition 7. *If $|\mathsf{B}(q)| < \infty$ and if the function $v \mapsto \partial_v \mathsf{B}(q)$ is lower semi-continuous, one has*

$$\mathcal{H}^{q,\mathsf{B}(q)} \left\{ x \mid \liminf_{r \searrow 0} \frac{\langle v, \varkappa(x,r) \rangle}{-\log r} < -\partial_v \mathsf{B}(q) \text{ for some } v \in \mathbb{E} \right\} = 0.$$

Proposition 8. *If, for some* q, $\mathcal{H}^{q,\mathsf{B}(q)}(\mathbb{X}) > 0$, *and if the function* $v \mapsto \partial_v \mathsf{B}(q)$ *is lower semi-continuous, then*

$$\dim \left\{ x \mid \liminf_{r \searrow 0} \frac{\langle v, \varkappa(x,r) \rangle}{-\log r} + \partial_v \mathsf{B}(q) \geq 0 \text{ for all } v \in \mathbb{E} \right\} \geq \mathsf{B}(q) - \partial_q \mathsf{B}(q).$$

Theorem 9. *If, for some* q, *the function* B *is differentiable with derivative* $\mathsf{B}'(q)$ *and if* $\mathcal{H}^{q,\mathsf{B}(q)}(\mathbb{X}) > 0$, *then one has* $\mathsf{b}(q) = \mathsf{B}(q)$ *and*

$$\dim X\big(-\mathsf{B}'(q)\big) = \mathrm{Dim}\, X\big(-\mathsf{B}'(q)\big) = \mathsf{B}^*\big(-\mathsf{B}'(q)\big).$$

Proof of Lemma 7

Take $\lambda > \partial_v \mathsf{B}(q)$ and $t > 0$ such that $\mathsf{B}(q+tv) < \mathsf{B}(q) + \lambda t$. Consider the set

$$F = \left\{ x \in \mathbb{X} \mid \liminf_{r \searrow 0} \frac{\langle v, \varkappa(x,r)\rangle}{-\log r} < -\lambda \right\}.$$

Given $\delta > 0$, for each $x \in F_n$, one can find $r_x > 0$ such that $r_x < \delta$ and $\langle v, \varkappa(x, r_x)\rangle - \lambda \log r_x \leq 0$.

Let $\emptyset \neq F' \subset F$. One can find (Besicovitch covering property) θ sequences $(x_{i,j})_j$ $(1 \leq i \leq \theta)$ of points of F' such that, for $i = 1, 2, \cdots, \theta$, the balls $\Big(B(x_{i,j}, r_{x_{i,j}})\Big)_j$ form a packing of F' and that these packings altogether form a cover of F'.

$$
\begin{aligned}
\overline{\mathcal{H}}_{\delta}^{q,\mathsf{B}(q)}(F') &\le \sum_{1\le i\le\theta}\sum_{j} \mathrm{e}^{-\left(\langle q,\varkappa(x_{i,j},r_{x_{i,j}})\rangle-\mathsf{B}(q)\log r_{x_{i,j}}\right)} \\
&\le \sum_{1\le i\le\theta}\sum_{j} \mathrm{e}^{-\left(\langle q+tv,\varkappa(x_{i,j},r_{x_{i,j}})\rangle-(\mathsf{B}(q)+\lambda t)\log r_{x_{i,j}}\right)} \\
&\le \theta\, \overline{\mathcal{P}}_{\delta}^{q+tv,\mathsf{B}(q)+\lambda t}(F').
\end{aligned}
$$

$$
\overline{\mathcal{H}}^{q,\mathsf{B}(q)}(F') \le \theta\, \overline{\mathcal{P}}^{q+tv,\mathsf{B}(q)+\lambda t}(F').
$$

If $F' = \bigcup F'_j$,

$$
\overline{\mathcal{H}}^{q,\mathsf{B}(q)}(F') \le \sum \overline{\mathcal{H}}^{q,\mathsf{B}(q)}(F'_j) \le \theta \sum \overline{\mathcal{P}}^{q+tv,\mathsf{B}(q)+\lambda t}(F'_j).
$$

$$
\overline{\mathcal{H}}^{q,\mathsf{B}(q)}(F') \le \theta\, \mathcal{P}^{q+tv,\mathsf{B}(q)+\lambda t}(F') = 0,
$$

$$
\mathcal{H}^{q,\mathsf{B}(q)}(F) = 0.
$$

Proof of Proposition 10

Set $X = \left\{ x \mid \liminf_{r \searrow 0} \frac{\langle v, \varkappa(x,r) \rangle}{-\log r} + \partial_v \mathsf{B}(q) \geq 0 \text{ for all } v \in \mathbb{E} \right\}$.

We have $\mathcal{H}^{q,\mathsf{B}(q)}(X) > 0$.

Take $\varepsilon > 0$. For $m \geq 1$, consider

$$F_{m,\varepsilon} = \left\{ x \in X \mid \langle q, \varkappa(x,r) \rangle - \left(\partial_q \mathsf{B}(q) + \varepsilon \right) \log r > 0 \text{ for } r \leq 1/m \right\}.$$

As $X = \bigcup_{m \geq 1} F_{m,\varepsilon}$, there exists m so that $\mathcal{H}^{q,\mathsf{B}(q)}(F_{m,\varepsilon}) > 0$.

Therefore, there exist m and a subset F of $F_{m,\varepsilon}$ such that $\overline{\mathcal{H}}^{q,\mathsf{B}(q)}(F) > 0$.

If $\Big\{B(x_j, r_j)\Big\}$ is a centered δ-cover of F, with $\delta < 1/m$, one has

$$
\begin{aligned}
\sum \mathsf{e}^{\left(\mathsf{B}(q)-\partial_q\mathsf{B}(q)-\varepsilon\right)\log r_j} &\geq \sum \mathsf{e}^{-\left(\langle q,\varkappa(x_j,r_j)\rangle-\mathsf{B}(q)\log r_j\right)} \\
&\geq \overline{\mathcal{H}}_{\delta}^{q,\mathsf{B}(q)}(F),
\end{aligned}
$$

which gives

$$
\mathcal{H}^{\mathsf{B}(q)-\partial_q\mathsf{B}(q)-\varepsilon}(F_{m,\varepsilon}) \geq \overline{\mathcal{H}}^{\mathsf{B}(q)-\partial_q\mathsf{B}(q)-\varepsilon}(F) \geq \overline{\mathcal{H}}^{q,\mathsf{B}(q)}(F) > 0.
$$

So, $\dim X \geq \dim F_{m,\varepsilon} \geq \mathsf{B}(q) - \partial_q\mathsf{B}(q) - \varepsilon$.

Gibbs and Frostman measures

Lemma 10. *If there exists a measure* $\mu^{[q]}$ *such that* $\limsup_{r \searrow 0} \frac{\mu^{[q]}(B(x,r))}{\mathrm{e}^{-(\langle q, \varkappa(x,r)\rangle - \mathrm{B}(q)\log r)}} < +\infty$ *for* $\mu^{[q]}$*-almost every* x*, then* $\mathcal{H}^{q,\mathrm{B}(q)}(\mathbb{X}) > 0$.

We call such a measure a Frostman measure at q.

When there exists a Borel measure $\mu^{[q]}$, and two positive numbers η and C such that, for all $x \in \mathbb{X}$, and for all $r \leq \eta$, one has

$$\frac{1}{C} \leq \frac{\mu^{[q]}(B(x,r))}{\mathrm{e}^{-(\langle q, \varkappa(x,r)\rangle - \mathrm{B}(q)\log r)}} \leq C$$

we say that $\mu^{[q]}$ is a Gibbs measure at q.

In [BMP], it was proven that the multifractal formula holds when Gibbs measures exist.

The Λ function

$$\overline{\mathcal{P}}_\delta^{q,t}(A) = \sup\left\{\sum_j \mathrm{e}^{-\left(\langle q,\varkappa(x_j,r_j)\rangle - t\log r_j\right)} \;\middle|\; \text{packing, } r_j \le \delta\right\}$$

$$\mathcal{P}_\delta^{*q,t}(A) = \sup\left\{\sum_j r_j^t \mathrm{e}^{-\langle q,\varkappa(x_j,r_j)\rangle} \;\middle|\; \text{packing, } \delta/2 < r_j \le \delta\right\}$$

$$\overline{\mathcal{P}}^{q,t}(A) = \lim_{\delta\searrow 0} \overline{\mathcal{P}}_\delta^{q,t}(A), \qquad \mathcal{P}^{*q,t}(A) = \lim_{\delta\searrow 0} \mathcal{P}_\delta^{*q,t}(A)$$

$$\Lambda(q) = \lim_{R\to+\infty} \inf\left\{t \mid \overline{\mathcal{P}}^{q,t}\Big(B(x_0,R)\Big) = 0\right\} \ge B(q)$$

Alternate definition:

$$\Lambda(q) = \lim_{R\to+\infty} \inf\left\{t \mid \mathcal{P}^{*q,t}\Big(B(x_0,R)\Big) = 0\right\}$$

When $\mathrm{e}^{\varkappa}$ is a measure and $\mathbb{X}$ is the boundary of an homogeneous tree, one gets the τ function of [BMP].

Theorems by Besicovitch and Eggleston

Theorem 11 (Besicovitch). *Let B_f be the set*

$$\left\{ x \in [0,1] \mid \limsup \frac{1}{n} \sum_{i=1}^{n} x_j \leq f \right\},$$

where $\sum x_j 2^{-j}$ is the dyadic expansion of x.
Then $\dim B_f = -f \log_2 f - (1-f) \log_2 (1-f)$ if $0 \leq f \leq 1/2$, and $\dim B_f = 1$ if $f \geq 1/2$.

Theorem 12 (Eggleston). *Let $f = (f_0, f_1, \ldots, f_{c-1})$ be a probability vector. Consider the set*

$$E_f = \{x \in [0,1] \mid \text{frequency of digit } j = f_j \text{ for } j = 0, 1, \ldots, c-1\}.$$

Then $\dim E_f = -\sum_{j=0}^{c-1} f_j \log_c f_j$.

Let c be an integer ≥ 2 and $\mathbb{X} = \{0, 1, 2, \ldots, c-1\}^{\mathbb{N}}$ endowed with the usual ultrametric distance: two sequences $(\varepsilon_n)_{n\geq 0}$ and $(\alpha_n)_{n\geq 0}$ are distant from c^{-k} if $\varepsilon_k \neq \alpha_k$ and if $\varepsilon_j = \alpha_j$ for all j such that $0 \leq j < k$.

If $x = (x_n)_{n\geq 0} \in \mathbb{X}$, set $\varphi_n(x, j) = \dfrac{1}{n}\text{card}\{0 \leq k < n \mid x_k = j\}$ for $j = 0, 1, \ldots, c-1$.

Let $p = (p_0, p_1, \cdots, p_{c-1})$ be a family of positive numbers. If $x = (x_n)_{n\geq 0} \in \mathbb{X}$, one sets

$$\varkappa(x, c^{-k}) = -\log \prod_{0\leq j<k} p_{x_j} = -k \sum_{0\leq j<c} \varphi_k(x, j) \log p_j.$$

and take $\mathbb{E}'$ to be $\mathbb{R}$..

It is easily seen that $\Lambda(q) = \log_c \sum_{0\leq j<c} p_j^q$.

If $q \in \mathbb{R}$, one sets, for $0 \le j < c$,

$$r_j = p_j^q \Big/ \sum_{0 \le k < c} p_k^q.$$

A measure $\mu^{[q]}$ is defined on $\mathbb{X}$ by the formula

$$\mu^{[q]}\Big(B(x, c^{-k})\Big) = \prod_{l=0}^{k-1} r_{x_l}.$$

It is easy to check that

$$\mu^{[q]}\Big(B(x, c^{-k})\Big) = \mathrm{e}^{-\Big(q\,\varkappa(x, c^{-k}) + k\Lambda(q)\log c\Big)}.$$

So, $\mu^{[q]}$ is a Gibbs measure. This implies $\mathcal{H}^{q,\Lambda(q)}(\mathbb{X}) > 0$, which has two consequences: $\mathsf{b}(q) = \mathsf{B}(q) = \Lambda(q)$ and the fact that the multifractal formalism holds for all q.

By taking $c = 2$ and $p = (1/2, 1)$, one gets the Besicovitch theorem.

By taking $p = (1/c, 1, \ldots, 1)$ one gets that the set of numbers of which the frequency of digit 0 in their base c expansion is f has

$$-f \log_c f - (1 - f) \log_c \frac{1 - f}{c - 1}$$

for its Hausdorff dimension.

Generalization

Let $p = \Big\{(p_{l,0}, p_{l,1}, \cdots, p_{l,c-1})\Big\}_{0\le l<\nu}$ be a family of positive numbers. If $x = (x_n)_{n\ge 0} \in \mathbb{X}$, one sets

$$\varkappa(x, c^{-k}) = \Big(-\log \prod_{0\le j<k} p_{l,x_j}\Big)_{0\le l<\nu}$$

and take $\mathbb{E}'$ to be $\mathbb{R}^\nu$.

It is easily seen that $\Lambda(q) = \log_c \Big(\sum_{0\le j<c} \prod_{0\le l<\nu} p_{l,j}^{q_l}\Big)$.

If $q \in \mathbb{R}^\nu$, one sets, for $1 \le j \le \nu$, $r_j = \prod_{0\le l<\nu} p_{l,j}^{q_l} \Big/ \sum_{0\le k<c} \prod_{0\le l<\nu} p_{l,k}^{q_l}$.
As previously, one considers the multinomial measure $\mu^{[q]}$ defined on $\mathbb{X}$ by the formula $\mu^{[q]}\Big(B(x, c^{-k})\Big) = \prod_{l=0}^{k-1} r_{x_l}$. As before, this is a Gibbs measure, which has two consequences: $\mathsf{b}(q) = \mathsf{B}(q) = \Lambda(q)$ and the fact that the multifractal formalism holds for all q.

Recalling the notation $\varphi_n(x,j) = \dfrac{1}{n}\,\mathrm{card}\{0 \le k < n \mid x_k = j\}$ for $j = 0, 1, \ldots, c-1$,

one has $\varkappa(x, c^{-k}) = \left(-k \sum_{0 \le j < c} \varphi_k(x,j) \log p_{l,j}\right)_{0 \le l < \nu}$.

Theorem 13. *Let $\nu < c$ and f_0, $f_1, \ldots,$ $f_{\nu-1}$ be positive numbers such that $\sum_{0 \le j < \nu} f_j \le 1$. Then,*

$$\dim\left\{x \in \mathbb{X} \mid \lim_{n \to +\infty} \varphi_n(x,j) = f_j \textit{ for } 0 \le j < \nu\right\}$$
$$= -\left(1 - \sum_{0 \le j < \nu} f_j\right) \log_c \frac{1 - \sum_{0 \le j < \nu} f_j}{c - \nu} - \sum_{0 \le j < \nu} f_j \log_c f_j.$$

Proof. Take $p_{j,j} = c^{-1}$ and $p_{l,j} = 1$ if $l \neq j$. Then

$$\Lambda(q) = \log_c \left(c - \nu + \sum_{0 \le j < \nu} c^{-q_j} \right)$$

and

$$\frac{\varkappa(x, c^{-k})}{k \log c} = \big(\varphi_k(x, j) \big)_{0 \le j < \nu}.$$

Then, it is easy to complete the computation of the Legendre transform.

Set $H_c(x_0, x_1, \ldots, x_{c-1}) = -\sum_{j=0}^{c-1} x_j \log_c x_j$.

Theorem 14. *Suppose $\nu < c$. Let $f_0, f_1, \ldots, f_{\nu-1}$ be non-negative numbers and consider the set*

$$B_f = \left\{ x \in \mathbb{X} \mid \limsup_{n\to\infty} \varphi_j(x,n) \leq f_j \text{ for } 0 \leq j < \nu \right\}.$$

Let $f_0^* \geq f_1^* \geq \cdots \geq f_{\nu-1}^*$ be the sequence $(f_j)_{0\leq j<\nu}$ rearranged in decreasing order, and $f_j^{**} = \sum_{j\leq k<\nu} f_k^*$. Then

1. *If $(c-\nu)f_0^* + f_0^{**} < 1$,*
   then $\dim B_f = H_c(f_0^*, \ldots, f_{\nu-1}^*, \frac{1-f_0^{**}}{c-\nu}, \frac{1-f_0^{**}}{c-\nu}, \ldots)$.

2. *For $0 \leq k < \nu - 1$, if $(c-\nu+k)f_k^* + f_k^{**} \geq 1$ and $(c-\nu+k+1)f_{k+1}^* + f_{k+1}^{**} < 1$,*
   then $\dim E = H_c(f_k^*, \ldots, f_{\nu-1}^*, \frac{1-f_{k+1}^{**}}{c-\nu+k+1}, \frac{1-f_{k+1}^{**}}{c-\nu+k+1}, \ldots)$.

3. *If $f_{\nu-1}^* \geq \frac{1}{c}$, then $\dim E = 1$.*

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