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β-transformations, Iterated Function Systems, and Cantor-type Sets

Qinghe Yin

Australian National University

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1. Introduction.

Question (M. Keane 1993): Let

$$\Lambda(\lambda) = \left\{ \sum_{k=1}^{\infty} i_k \lambda^k : \; i_k = 0, 1, 3 \right\}$$

Is the Hausdorff dimension $\dim_H(\Lambda(\lambda))$ continuous for $\lambda \in [\frac{1}{4}, \frac{1}{3}]$?

Answer (Pollicott & Simon 1995): No, it is not continuous.

$$\dim_H(\Lambda(\lambda)) \begin{cases} = \frac{\log 3}{-\log \lambda} & \text{for almost all } \lambda \in [\frac{1}{4}, \frac{1}{3}] \\ < \frac{\log 3}{-\log \lambda} & \text{for a dense set.} \end{cases}$$

Note: In $\Lambda(\lambda)$, there is no restriction on digits 0, 1, 3. Different series may express the same number.

New setting: Let

$$C_{\beta;013} = \overline{\left\{ x \in [0,1] : \text{ the } \beta\text{-expansion of } x \text{ is } \sum_{k=1}^{\infty} \frac{i_k}{\beta^k}, \; i_k \in \{0, 1, 3\} \right\}}$$

Question: Is $\dim_H(C_{\beta;013})$ continuous for $\beta \in (3, 4]$?

Answer: Yes.

$$\dim_H(C_{\beta;013}) = \frac{\log \alpha_{\beta;013}}{\log \beta}$$

where $\alpha_{\beta;013}$ continuously depends on β.

2. β-transformation and β-expansion.

Fix $\beta > 1$. Given $x \in [0, 1)$, we can express x as

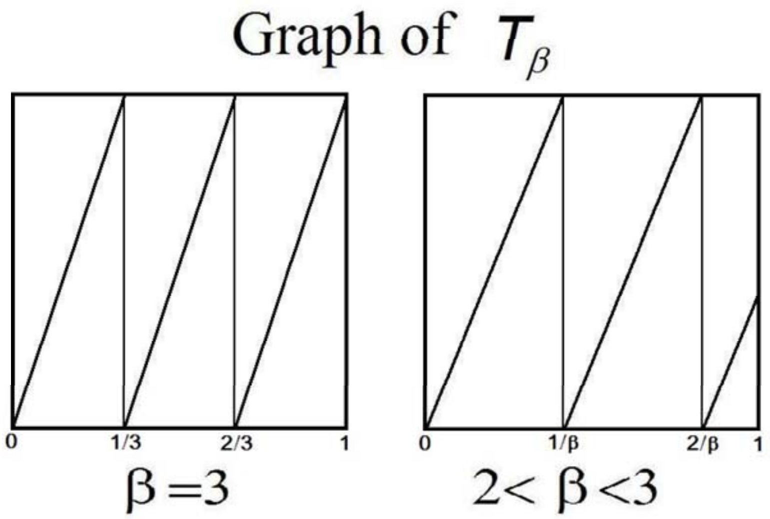
$$x = \sum_{n=1}^{\infty} \frac{a_n}{\beta^n} \tag{1}$$

where $a_i \in \{0, 1, \cdots, \lceil \beta \rceil - 1\}$.

If β is not an integer, we may have continuum (or uncountably) many such series for some x. We need a rule on how to choose the digits a_n to get a unique series for each x.

Definition. *Given $\beta > 1$, the β transformation T_β on $[0, 1)$ is defined by*

$$T_\beta x = \beta x - \lfloor \beta x \rfloor.$$



Now we have $x = \frac{\lfloor \beta x \rfloor}{\beta} + \frac{T_\beta x}{\beta}$. Let $a_k = \lfloor \beta T_\beta^{k-1} x \rfloor$, for $k = 1, \cdots, n$. Then

$$x = \frac{a_1}{\beta} + \cdots + \frac{a_n}{\beta^n} + \frac{T_\beta^n x}{\beta^n}.$$

Choosing digits in this way, we say the series (1) is the β-expansion or **greedy** β-expansion of x.

Let

$$1 = \frac{e_1}{\beta} + \frac{e_2}{\beta^2} + \cdots$$

be the β-expansion of 1, where $e_1 = \lfloor \beta \rfloor$. β is **simple**, if the β-expansion of 1 has only finite many non-zero terms. Let

$$(\epsilon_1, \epsilon_2, \cdots) = (e_1, e_2, \cdots)$$

if β is non-simple, or

$$(\epsilon_1, \epsilon_2, \cdots) = (e_1, e_2, \cdots e_{k-1}, e_k - 1)^{\mathbb{N}}$$

if $e_k \neq 0$ but $e_i = 0$ for $i > k$.

Theorem 1 (Parry 1961) *The series* $\sum_{n=1}^{\infty} \frac{a_n}{\beta^n}$ *is the* β*-expansion of some* $x \in [0, 1)$ *if and only if*

$$(a_n, a_{n+1}, \cdots) < (\epsilon_1, \epsilon_2, \cdots) \qquad (2)$$

We say $(a_1, a_2, \cdots)$ is β**-admissible** if (2) holds. We have

$$x < y \Leftrightarrow (a_1, a_2, \cdots) < (b_1, b_2, \cdots)$$

where $x = \sum_{n=1}^{\infty} \frac{a_n}{\beta^n}$ and $y = \sum_{n=1}^{\infty} \frac{b_n}{\beta^n}$.

Theorem 2 (Parry 1961) T_β *possesses an invariant measure* μ_β *with density function* $\frac{1}{\gamma} h_\beta(x)$ *where*

$$h_\beta(x) = \sum_{x < T_\beta^n 1} \frac{1}{\beta^n}$$

and $\gamma = \int_0^1 h_\beta(x) dx$. *Furthermore,* T_β *is ergodic with respect to* μ_β.

We know that when β is an integer, $\mu_\beta = \lambda$, the Lebesgue measure on $[0, 1)$.

Let

$$\Sigma_m = \{0, 1, \cdots, m-1\}^{\mathbb{N}}$$

where $m = \lceil \beta \rceil$, and Σ_β be the closure of

$$\left\{ (a_1, a_2, \cdots) \in \Sigma_m | \sum_{n=1}^{\infty} \frac{a_n}{\beta^n} \text{ is } \beta\text{-expansion for some } x \right\}$$

where the closure is according to the product topology. Then T_β is corresponding to the left shift transformation on Σ_β. We also call Σ_β the β-shift.

Example. Let $\beta = \frac{1+\sqrt{5}}{2} = g$. The g-expansion of 1 is

$$1 = \frac{1}{g} + \frac{1}{g^2}.$$

Then g is simple. We have

$$h_g(x) = \begin{cases} 1 + \frac{1}{g} = g & 0 \leq x < \frac{1}{g} \\ 1 & \frac{1}{g} \leq x < 1 \end{cases}$$

Density function

$$\rho_g(x) = \begin{cases} \frac{g^3}{1+g^2} & 0 \leq x < \frac{1}{g} \\ \frac{g^2}{1+g^2} & \frac{1}{g} \leq x < 1 \end{cases}$$

$$\Sigma_g = \{(a_1, a_2, \cdots) | a_n \in \{0, 1\}, \text{ but } a_n a_{n+1} \neq 11\}$$

Σ_g is the **golden-mean shift**. Σ_g is a shift of finite type determined by the forbidden word 11.

In general, Σ_β is a shift of finite type if and only if β is simple.

3. Iterated Function Systems.

An Iterated Function System (IFS) is an $(n+1)$-tuple

$$(X; f_0, f_1, \cdots, f_{n-1})$$

where X is a compact metric space and $f_i : X \mapsto X$ is a contractive map such that

$$d(f_i(x), f_i(y)) \le r_i d(x, y), \text{ for some } 0 < r_i < 1.$$

Theorem (Hutchinson 1981) *Let $(X, f_0, \cdots, f_{n-1})$ be an IFS. Then there exists a unique compact subset E such that*

$$E = f_0(E) \cup f_1(E) \cup \cdots \cup f_{n-1}(E).$$

We call E the **attractor** of the IFS.

Open Set Condition. We say $(X; f_0, \cdots, f_{n-1})$ has an open set condition, if there exists an open set O such that

$$f_i(O) \subset O \quad \text{and} \quad f_i(O) \cap f_j(O) = \emptyset, \quad i \neq j.$$

Theorem (Hutchinson 1981) *Let $(X; f_0, \cdots, f_{n-1})$ be an IFS such that*

$$d(f_i(x), f_i(y) = r_i d(x, y).$$

With an open set condition, the Hausdorff dimension of the attractor E is

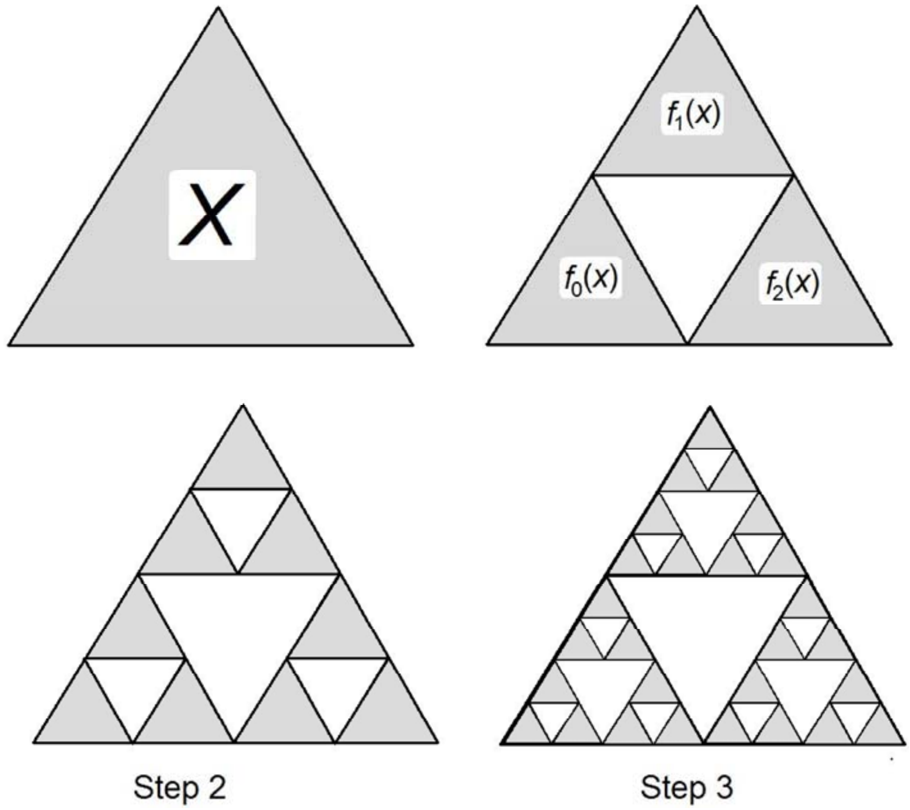
$$\dim(E) = s$$

where s is the root of

$$r_0^s + r_1^s + \cdots + r_{n-1}^s = 1$$

The attractor E in above theorem is a **self-similar** fractal.

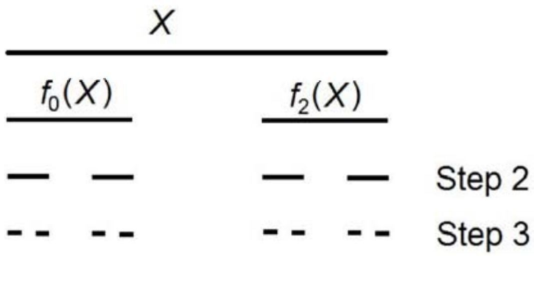
Example 1. Sierpinski Triangle



The Hausdorff dimension of the Sierpinski triangle is $\frac{\log 3}{\log 2} = 1.5849\cdots$

Example 2. Cantor middle-third set

$X = [0,1]$, $f_0(x) = \frac{x}{3}$, $f_2(x) = \frac{x+2}{3}$.



The Hausdorff dimension of the Cantor middle-third set is

$$\dim(C) = \frac{\log 2}{\log 3} = 0.6309\cdots$$

Let E be the attractor of $(X; f_0, f_1, \cdots, f_{n-1})$. For any sequence $(i_1, i_2, \cdots) \in \Sigma_n$, we have

$$\lim_{k\to\infty} f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_k}(y) \in E.$$

The above limit exists for any $y \in X$ and is independent of y.

On the other hand, for each $x \in E$, there exists at least one sequence $(i_1, i_2, \cdots) \in \Sigma_n$, such that

$$x = \lim_{k\to\infty} f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_k}(y).$$

Other Type of Attractors –Subsets of the Attractor E

Markov Attractor. Given an $n \times n$ (0,1)-matrix M, assume that M is irreducible. Then there exist compact subsets $E_0, \cdots, E_{n-1}$, such that

$$E_i = \bigcup_{M_{ij}=1} f_i(E_j).$$

We call $(E_0, \cdots, E_{n-1})$ or $E_M = \bigcup_{i=0}^{n-1} E_i$ the **Markov attractor** associated with M.

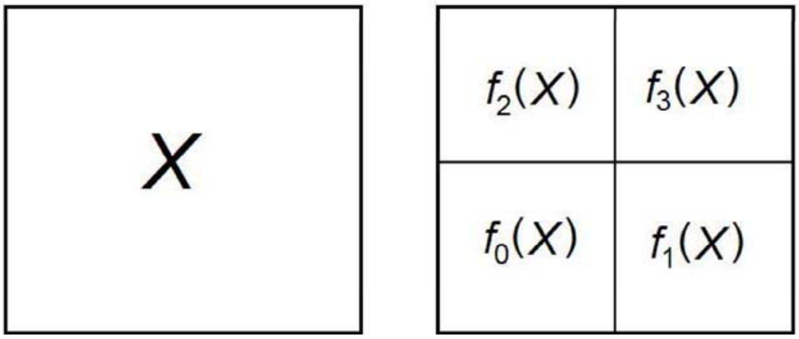
When all the maps are similitudes, with an open set condition, the Hausdorff dimension of E_i or E_M is give by

$$||MR^s|| = 1,$$

where $||\cdot||$ stands for the Perron-Frobenus eigen-value of the given matrix and $R^s = \begin{pmatrix} r_0^s & & \\ & \ddots & \\ & & r_{n-1}^s \end{pmatrix}$.

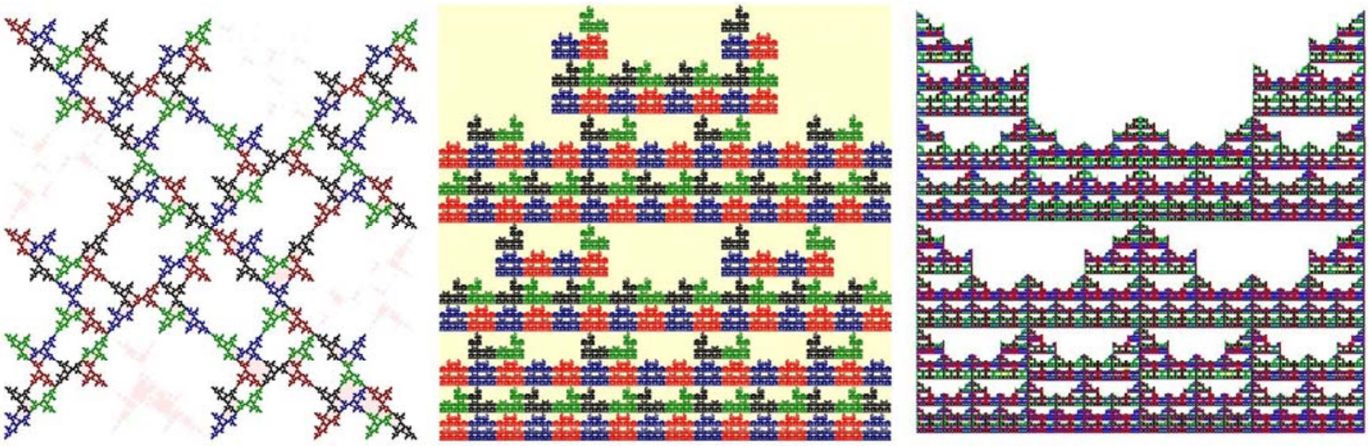
When all the entries of M are 1, the $E_M = E$ is the attractor.

Example. The IFS is as shown in the graph.



Let

$$M_1 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$



From left to right: Markov attractors associated with M_1, M_2, M_3.

$$\dim(E_1) = \frac{\log 3}{\log 2} = 1.5849\cdots$$

$$\dim(E_2) = \dim(E_3) = \frac{\log(3+\sqrt{17})}{\log 2} = 1.8325\cdots$$

β-attractor. Given $\beta \leq n$, then the β-shift Σ_β is a subspace of Σ_n. Let E_β be the set of all $x \in X$ such that there exists a sequence $(i_1, i_2, \cdots) \in \Sigma_\beta$ such that

$$x = \lim_{k\to\infty} f_{i_1} \circ f_{i_2} \circ \cdots f_{i_k}(y).$$

We say E_β is the **β-attractor.**

If β is an integer then E_β is the attractor of some IFS. For $\beta = g$, E_g is the Markov attractor associated with

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

In general, if β is simple then E_β is a Markov attractor for some IFS.

Examples. Given IFS $(X; f_0, f_1)$, assume that

$$1 = \frac{1}{\beta} + \frac{1}{\beta^4}.$$

Then

$$\Sigma_\beta = \{(i_1, i_2, \cdots)|i_k \in \{0,1\}, (i_k, i_{k+1}, \cdots) < (1,0,0,1,\cdots), \forall k\}.$$

Σ_β is determined by the forbidden words $(1,0,0,1,(1,0,1)$ and $(1,1)$. Consider the following maps:

$$f_0 \circ f_0 \circ f_0, \quad f_0 \circ f_0 \circ f_1, \quad f_0 \circ f_1 \circ f_0, \quad f_1 \circ f_0 \circ f_0.$$

Let

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Then E_β is the Markov attractor associated with M.

Theorem 1. *Let* $(X; f_0, \cdots, f_{n-1})$ *be an IFS such that*

$$d(f_i(x), f_i(y)) = rd(x, y)$$

for all $x, y \in X$ *and all* $0 \leq i \leq n-1$, *where* $0 < r < 1$. *Given* $\beta < n$, *let* E_β *be the* β*-attractor. Assume that*

$$f_i(E_\beta) \cap f_j(E_\beta) = \emptyset, \quad \textit{if } i \neq j.$$

Then the Hausdorff dimension of E_β *is given by*

$$\dim(E_\beta) = \frac{\log \beta}{-\log r}.$$

Notice that if β is an integer then the above result is a special case of the Hutchinson's theorem. The above theorem extends Hutchinson's theorem to IFS with non-integer many (β) maps in a special case.

4. Cantor-type sets with β-expansions.

Fix $\beta > 2$ and $j_i \in \{0, 1, \cdots, \lceil\beta\rceil - 1\}$ with

$$0 \leq j_0 < j_1 < \cdots < j_{m-1}.$$

Define

$$\frac{C_{\beta;j_0 j_1 \cdots j_{m-1}}}{} = \left\{ x \in [0,1] : \text{ the } \beta\text{-expansion of } x \text{ is } \sum_{k=1}^{\infty} \frac{i_k}{\beta^k},\ i_k \in \{j_0, \cdots, j_{m-1}\} \right\}$$

Let $3 < \beta \leq 4$, $j_0, j_1, j_2 = 0, 1, 3$. Then $C_{\beta;013}$ is the set we defined earlier.

Theorem 2. *The Hausdorff dimension of the Cantor-type set with β-expansion, $C_{\beta;j_0\cdots j_{m-1}}$, is given by*

$$\dim(C_{\beta;j_0\cdots j_{m-1}}) = \frac{\log \alpha_{\beta;j_0\cdots j_{m-1}}}{\log \beta} \tag{3}$$

where $\alpha_{\beta;j_0\cdots j_{m-1}} \leq m-1$ and is continuously depends on β.

Theorem 2 answers the question asked at the beginning.

Sketch of Proof. Consider the IFS $(X; f_0, \cdots, f_{m-1})$, where $X = \left[0, \frac{\beta}{\beta-1}\right]$, $f_i(x) = \frac{x+j_i}{\beta}$. Then $C_{\beta;j_0\cdots j_{m-1}}$ is the α-attractor of this IFS, for $\alpha = \alpha_{\beta;j_0\cdots j_{m-1}}$.

If the disjoint condition holds (e.g., $C_{\beta;02}$) then we get (3) from Theorem 1. Unfortunately, the disjoint condition not always holds (e.g., $C_{\beta;013}$, for some β).

Let $\alpha' < \alpha_{\beta;j_0\cdots j_{m-1}}$. Apply Theorem 1 to the α'-attractor which is a subset of the α-attractor and has the disjoint condition. Then

$$\dim(C_{\beta;j_0\cdots j_{m-1}}) \geq \frac{\log \alpha'}{\log \beta}.$$

How $\alpha_{\beta;j_0\cdots j_{m-1}}$ is Calculated?

Assume $2 < \beta \leq 3$ and take $C_{\beta;02}$ as an example.

Let $z = \max\{x \in C_{\beta;02}\}$. Suppose that

$$z = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots$$

where $a_i \in \{0, 2\}$. Define $b_i = a_i/2$. Then $\alpha_{\beta;02}$ is determined by

$$1 = \frac{b_1}{\alpha} + \frac{b_2}{\alpha^2} + \cdots .$$

Example 1. Let $\beta = 3$. Then

$$z = 1 = \frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \cdots$$

$a_i = 2$. Then $b_i = 1$. Let $1 = \frac{1}{\alpha} + \frac{1}{\alpha^2} + \cdots$. Then $\alpha = 2$. Hence $\alpha_{3;02} = 2$. By theorem 2,

$$\dim(C_{3;02}) = \frac{\log 2}{\log 3}.$$

Example 2. Let $1 = \frac{2}{\beta} + \frac{1}{\beta^2} + \frac{2}{\beta^3}$. Then $\beta = 2.658\cdots$. The maximum of E_2 is given by

$$a = \frac{2}{\beta} + \frac{2}{\beta^3} + \frac{2}{\beta^5} + \cdots$$

Then $\alpha(\beta)$ is given by

$$1 = \frac{1}{\beta} + \frac{1}{\beta^3} + \frac{1}{\beta^5} + \cdots$$

which gives $\alpha = \frac{1+\sqrt{5}}{2} = g = 1.618\cdots$. Thus

$$\dim(C_{\beta;02}) = \frac{\log g}{\log \beta} = 0.49\cdots$$

Actually, for all $\beta \in [1+\sqrt{2}, 1+\sqrt{3}]$ we have $\alpha_{\beta;02} = g$. Hence $\dim(C_{\beta;02})$ is decreasing for β in this interval. However, we have

$$\lim_{\beta \to 2} \dim(C_{\beta;02}) = 0$$

and

$$\lim_{\beta \to 3} \dim(C_{\beta;02}) = \dim(C_{3;02}) = \frac{\log 2}{\log 3}.$$

So, in general, we have an increasing trend.

Theorem 3. *For almost all $\beta \in (2,3)$, we have*

$$\frac{d \dim_H(C_{\beta;02})}{d\beta} < 0.$$

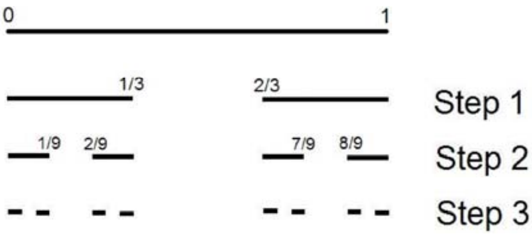
For other points, the derivative or one-sided derivative is ∞. These points form a Cantor-type set.

Construct $C_{\beta;02}$, for $2 < \beta \leq 3$ in Different Ways.

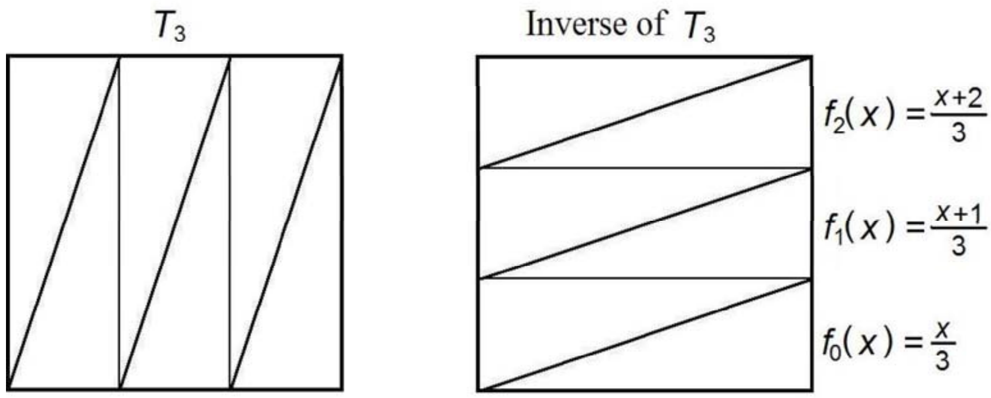
We can construct the Cantor middle-third set in three ways:

1. $C = C_{3;02}$.

2. Delete the middle-third open intervals.



3. Consider the inverse functions of $T_3 x = 3x \pmod 1$.

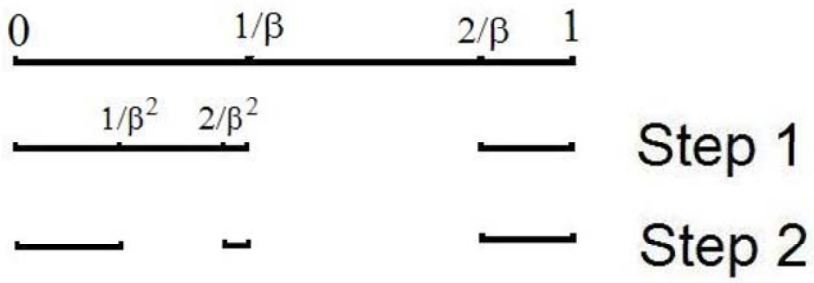


Then C is the attractor of the IFS $([0,1], f_0, f_2)$.

Taking $2 < \beta < 3$, we imitate the above constructions:

1. $C_{\beta;02}$.

2. Interval deleting.

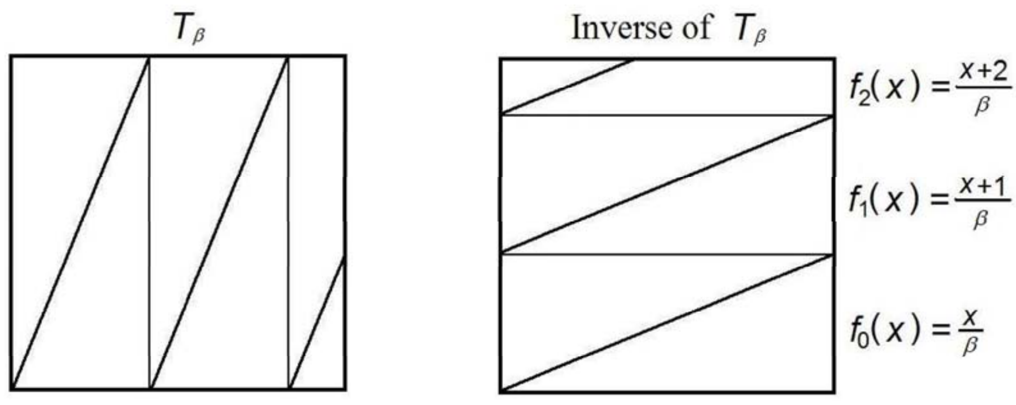


At k-th step we delete the open intervals

$$\left(\frac{a_1}{\beta} + \cdots + \frac{a_{k-1}}{\beta^{k-1}} + \frac{1}{\beta^k}, \frac{a_1}{\beta} + \cdots + \frac{a_{k-1}}{\beta^{k-1}} + \frac{2}{\beta^k}\right).$$

We use E_1 to denote the set constructed in this way.

3. Consider the inverse of $T_\beta x = \beta x \pmod 1$.



Again we take f_0 and f_2. This time f_2 is not defined on the whole space $[0, 1]$, but only on $[0, \beta - 2]$. So $([0, 1]; f_0, f_2)$ is not an IFS in original sense. For this "IFS" we also define its "attractor" E_2 as a compact set such that

$$E_2 = f_0(E_2) \cup f_2(E_2 \cap [0, \beta - 2]).$$

4. Define $\tilde{f}_0 = f_0$ and $\tilde{f}_2(x) = \begin{cases} \frac{x+2}{\beta} & x \in [0, \beta - 2], \\ 1 & \beta - 2 < x \le 1. \end{cases}$ Then we have an IFS $([0,1]; \tilde{f}_0, \tilde{f}_2)$. Use E_3 to denote the attractor of this IFS.

Questions:

1. Does E_2 exist? If so, is it unique?
2. Are $C_{\beta;02}, E_1,\ E_2,\ E_3$ equal, or what are their relations?
3. Do they have the same Hausdorff dimensions?

The following theorem answer these questions 1 and 2.

Theorem 4. *Let $C_{\beta;02}, E_1,\ E_2,\ E_3$ be defined as in the above. Then they have the following relations:*

1. $C_{\beta;02} \subseteq E_1$.

2. E_2 exists. It can be either $C_{\beta;02}$ or E_1.

3. $E_3 = E_1$.

4. If $E_1 \neq C_{\beta;02}$, then $E_1 \backslash C_{\beta;02}$ contains countably many isolated points.

Thus they all have the same Hausdorff dimension.

Further Research:

1. The Hausdorff dimension for β-attractor (Theorem 1) is calculated under a disjoint condition. I believe it is true under an open set condition. But I even haven't found a natural way to define the open set condition in this case.

2. It is a challenge to obtain a formula for the Hausdorff dimension when the maps have different contractive ratios.

3. It is natural to study the Hausdorff measure for β-attractors and Cantor type sets constructed by β-expansions of their dimension.

THE END