MATH 402 Homework 10

Due Friday December 1, 2017

Exercise 1. (a) Suppose that $z = x + iy$ is a point of the complex plane corresponding to the point $P = (X, Y, Z)$ of the unit sphere under stereographic projection. Prove that

$$X = \frac{2x}{x^2 + y^2 + 1}, \quad Y = \frac{2y}{x^2 + y^2 + 1}, \quad \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}.$$

(Hint: recall that the line ℓ between two points A and B has the form $\{tA + (1 - t)B \mid t \in \mathbb{R}\}$.)

(b) Conversely, show that $x = \frac{1}{1-Z}X$ and $y = \frac{1}{1-Z}Y$.

Exercise 2.

(a) Let $f(z) = \frac{az-b}{cz-d}$ be a Möbius transformation, so $ad - bc \neq 0$. Prove that we can find different complex numbers a', b', c', d' so that

$$f(z) = \frac{a'z - b'}{c'z - d'}$$

and $a'd' - b'c' = 1$.

(b) Show that the set $\mathcal{U}$ of Möbius transformations that preserve the unit disk is a group.

Exercise 3. Fix α with $|\alpha| < 1$. In this exercise, we consider the Möbius transformation given by $T(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$, and show why it makes sense to think of this as translation along the line ℓ passing through 0 and α.

(a) Show that T has exactly two fixed points, which are both on the boundary of the unit disk. In particular, it has no fixed point inside the Poincaré disk.

(b) Show that T preserves the line $\ell = \{t\alpha | t \in \mathbb{R}\}$. (Note that this is the Euclidean line through 0 and α, but also restricts to the diameter corresponding to the hyperbolic line through 0 and α.) Show that the fixed points from part (a) are the omega points of this line.

(c) Show that for any Poincaré point $t\alpha$ on ℓ, the Poincaré distance from $t\alpha$ to $T(t\alpha)$ is always the same.

Exercise 4. In this exercise we will see that inversion with respect to the circle defining a Poincaré line is the same as the hyperbolic reflection across that line in the Poincaré model.

Let ℓ be a Poincaré line. Define a map f on the Poincaré disk by $f(P) = P_0$, where P_0 is the inverse point to P with respect to the circle c_ℓ on which ℓ is defined.

(a) Use results on circle inversion to show that f maps the Poincaré disk to itself.

(b) Prove that f is an isometry of the Poincaré disk (i.e. prove that it preserves the Poincaré distance function).

(c) Finally, show that f must be a reflection about the line ℓ.

Remember that in addition to the points assigned to each question, you will receive up to five further points for neatness and organization.