<u>Recall</u> Given any $C, D \in \operatorname{Div} X$, $\exists$ non-singular very ample curves C', D', E', F'

s.t. ① C', E' meet D', F' transversally

② $C \sim C' - E'$, $D \sim D' - F'$

Then we set $(**)$ $C.D = \# C' \cap D' - \# C' \cap F' - \# E' \cap D' + \# E' \cap F'$

<u>Theorem</u> $(**)$ defines a unique pairing $\operatorname{Div} X \times \operatorname{Div} X \to \mathbb{Z}$ $(C, D) \mapsto C.D$

s.t. ① if C, D are irreducible non-singular curves meeting transversally

$$C.D = \# C \cap D$$

② $C.D = D.C$

③ $(C_1 + C_2).D = C_1.D + C_2.D$

④ $C_1 \sim C_2 \Rightarrow C_1.D = C_2.D$

<u>Remarks on proof</u> (1-4) are equivalent to the formula $(**)$, from which uniqueness follows.

So it remains to check that $(**)$ is well-defined.

<u>Key observation</u> Let C be ~~any~~ an irreducible non-singular curve on X, and let D be any curve meeting C transversally.

Then $\# C \cap D = \deg_C (\mathcal{L}(D) \otimes \mathcal{O}_C)$.

<u>proof</u> SES: $0 \to \mathcal{L}(-D) \otimes \mathcal{O}_C \to \mathcal{O}_C \to \mathcal{O}_{C \cap D} \to 0$.

$\Rightarrow \mathcal{L}_C(C \cap D) = \mathcal{L}(D) \otimes \mathcal{O}_C$

$\Rightarrow$ (since C intersects D transversally) $\deg_C \mathcal{L}(D) \otimes \mathcal{O}_C = \# C \cap D$ $\square$

<u>Nice property</u>: (Matches our geometric intuition; allows us to compute $C.D$ without moving the curves.)

If C, D are curves on X having no common irreducible component,

$$C.D = \sum_{P \in C \cap D} (C.D)_P$$

$\triangleq \dim \mathcal{O}_{X,P}/(f,g)$

<u>proof</u> <u>Step 1</u> SES $0 \to \mathcal{L}(-D) \otimes \mathcal{O}_C \to \mathcal{O}_C \to \mathcal{O}_{C \cap D} \to 0$

- $C \cap D$ is a scheme supported at $P \in C \cap D$, with structure sheaf $\mathcal{O}_{X,P}/(f,g)$

$$\Rightarrow \dim H^0(\mathcal{O}_{C\cap D}) = \sum_{P\in C\cap D} (C.D)_P$$

<u>Step 2</u>

$$0 \to H^0(\mathcal{L}(-D)\otimes\mathcal{O}_C) \to H^0(\mathcal{O}_C) \to H^0(\mathcal{O}_{C\cap D}) \to$$
$$\to H^1(\mathcal{L}(-D)\otimes\mathcal{O}_C) \to H^1(\mathcal{O}_C) \to H^1(\mathcal{O}_{C\cap D}) = 0.$$

$$\Rightarrow \chi(\mathcal{L}(-D)\otimes\mathcal{O}_C) - \chi(\mathcal{O}_C) + \underbrace{\dim H^0(\mathcal{O}_{C\cap D})}_{=\sum_P (C.D)_P} = 0$$

$\therefore$ $\dim H^0(\mathcal{O}_{C\cap D})$ depends only on the linear equivalence class of D.

likewise, only on the linear equivalence class of C.

So we can replace $C = C'-E'$, $D = D'-F'$, curves meeting transversally

<u>Step 3</u>: Expanding both sides, it is enough to show that $C.D = \sum_{P\in C\cap D}(C.D)_P$ when C, D are non-singular curves meeting transversally.

Then $(C.D)_P = 1$ $\forall P\in C\cap D$, so

$$\sum_{P\in C\cap D}(C.D)_P = \# C\cap D \text{ as claimed.} \qquad \square$$

<u>Definition</u> for $D\in \mathrm{Div}X$, $D^2 = D.D$ is the self-intersection number of D

e.g. for K the canonical divisor, K^2 is an invariant of the surface

$(\mathcal{L}(K) = \omega_X = \Lambda^2\Omega_{X/k}.)$

<u>Rmk</u>: even for C a non-sing. curve on X, we cannot use this prop to calculate C^2.

but: • $C^2 = \deg(\mathcal{L}(C)\otimes\mathcal{O}_C)$.

• the ideal sheaf of $\mathcal{I}$ of C on X is $\mathcal{L}(-C)$ as

$$\Rightarrow \mathcal{I}/\mathcal{I}^2 \cong \mathcal{L}(-C)\otimes\mathcal{O}_C \qquad 0\to\mathcal{L}(-C)\to\mathcal{O}_X\xrightarrow{\pi}\mathcal{O}_C$$
$$f \longmapsto f\otimes 1.$$
$$\left(fg\in\mathcal{I}^2 \longmapsto fg\otimes 1 = f\otimes\pi(g)1 = 0.\right)$$

$$\Rightarrow \mathcal{L}(C)\otimes\mathcal{O}_C \cong \mathcal{N}_{C/X} = \mathcal{H}om(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_C) \quad \text{(normal sheaf)}$$

so $C^2 = \deg \mathcal{N}_{C/X}$.

Example A $X = \mathbb{P}^2$, so $\text{Pic}\, X = \mathbb{Z}$, generated by the class h of a line.

- $h^2 = \{x\text{-axis}\} \cdot \{y\text{-axis}\} = 1$

  ↳ by linearity, this determines the intersection pairing on X:

  eg. if C, D are curves of deg n, m, then $C \sim nh$, $D \sim mh$, and so $C.D = nm h^2 = nm$. (Bézout).

- $K = -3h$, so $K^2 = 9$

Example B X = non-singular quadric surface Q in $\mathbb{P}^3$; $\text{Pic}\, X = \mathbb{Z} \oplus \mathbb{Z}$.

- take as generators lines ℓ of type $(1,0)$ and m of type $(0,1)$

  ↳ $\ell^2 = 0$, $m^2 = 0$, $\ell m = 1$

  - by linearity: if C has type (a,b) and D has type (a',b'), $C.D = ab' + a'b$.

- K has type $(-2,-2)$, so $K^2 = 8$.

Proposition (Adjunction Formula)

If C is a non-singular curve of genus g on X, then $2g-2 = C.(C+K)$

proof (II. 8.20) $\Rightarrow$ $\omega_C \cong \omega_X \otimes \mathcal{L}(C) \otimes \mathcal{O}_C$

Now $\deg \omega_C = 2g-2$.

and $\deg(\omega_X \otimes \mathcal{L}(C) \otimes \mathcal{O}_C) = \deg_C(\mathcal{L}(K+C) \otimes \mathcal{O}_C) = (C+K).C$ $\square$

Examples: we can use this to compute the ~~degree~~ genus of $C \subset X$.

A) $C \subset \mathbb{P}^2$ curve of deg d. $\Rightarrow$ $C \sim dh$

so $2g-2 = dh(dh-3h) = d(d-3)h^2 = d(d-3)$

$\Rightarrow 2g = d^2 - 3d + 2 = (d-1)(d-2)$.

B) C of type (a,b) on $X = Q$, $\Rightarrow$ $C+K$ has type $(a-2, b-2)$

$\therefore 2g-2 = C.(C+K) = a(b-2) + b(a-2) = 2ab - 2a - 2b$

$\Rightarrow g = ab - a - b + 1$

(done as an example in lecture, 21 April)

$\boxed{1.5}$

(a) if X is a surface of degree d in $\mathbb{P}^3$, then $K^2 = d(d-4)^2$.

$X \subset \mathbb{P}^3$ with hyperplane section H

$\omega_X \cong \mathcal{O}_X(d-4) \Rightarrow K \sim (d-4)H$

so $K^2 = (d-4)^2 H^2 = (d-4)H$ (by Ex 1.2)

(b) if X is a product of non-singular curves C, C' of genus g, g' respectively, then $K^2 = 8(g-1)(g'-1)$

$\omega_{C\times C'} \cong p_1^*\omega_C \otimes p_2^*\omega_{C'}$ so $K_X = p_1^*K_C + p_2^*K_{C'}$

and $K_X^2 = (p_1^*K_C)^2 + 2(p_1^*K_C)(p_2^*K_{C'}) + (p_2^*K_{C'})^2$

Now if $\delta = \sum a_i P_i$ is any divisor on C,

$p_1^*\delta = \sum a_i \{P_i\} \times C'$

Claims 1) $\forall P, Q \in C$, $(\{P\}\times C')(\{Q\}\times C') = 0$

$\forall P, Q \in C'$ $(C\times\{P\})(C\times\{Q\}) = 0$

2) $\forall P \in C, Q \in C'$, $(\{P\}\times C')(C\times\{Q\}) = 1$

To show 1) note that if $P \neq Q$, the curves don't intersect, so we get 0

if $P = Q$, then we have adjunction:

$2g'-2 = C'_P.(C'_P + K_X)$

$= C_P'.C_P' + \sum_Q (C_P'.C_Q') + \sum_{Q\in K_{C'}} (C_P'.C_Q)$

$= C_P'.C_P' + 0 + (2g'-2)$ ← using claim 2

(assuming $g \geq 2$, K_C is basepoint free, so we can choose a realisation that doesn't involve P at all)

$\Rightarrow C_P'.C_P' = 0$

2) it is clear that $(\{P\}\times C'), (C\times\{Q\})$ intersect 1 time, transversally

So $(p_1^*\delta)^2 = 0$, $(p_2^*\delta)^2 = 0$. $\forall \delta$

and $K_X^2 = 2(p_1^*K_C)(p_2^*K_{C'}) = 2(2g-2)(2g'-2)$

$= 8(g-1)(g-2)$ as claimed