Chiral algebras, factorization algebras, and Borcherds's "singular commutative ring" approach to vertex algebras

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Section 1

Motivation

Context

vertex algebras chiral algebras factorization algebras

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vertex algebras

chiral algebras ⟷ factorization algebras

Koszul duality

[BD]; [FG]

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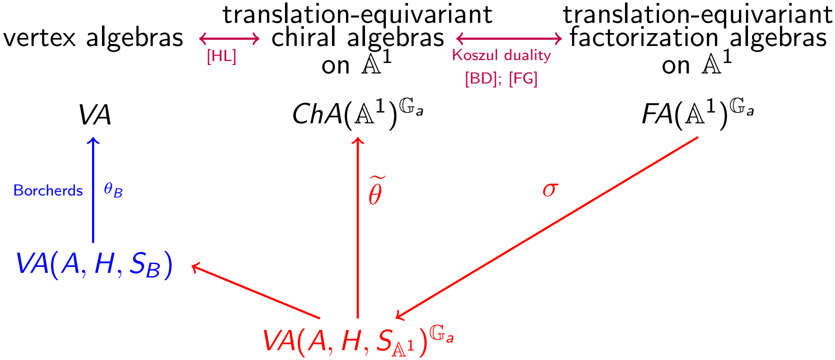
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A. Motivating questions (Borcherds)

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- How far is θ_B from being an equivalence? Can we construct examples of well-known vertex algebras in the category $VA(A, H, S_B)$ and understand their structure in that category?

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Can we use a geometric approach to understand this better?

C. Motivating question

Can we adapt Borcherds' definition of a quantum (A, H, S)-vertex algebra to the geometric setting?

D. Motivating definitions - chiral algebras

A chiral algebra on X is a right $\mathcal{D}$-module $\mathcal{A}_X$ on X equipped with a Lie bracket

$$\mu^{ch} : j_* j^* (\mathcal{A}_X \boxtimes \mathcal{A}_X) \to \Delta_! \mathcal{A}_X \in \mathcal{D}(X \times X).$$

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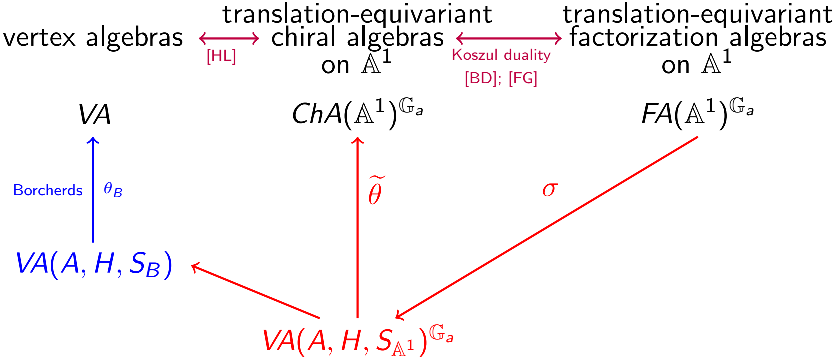
2. **Factorization isomorphisms.**
   e.g. $c : j^*(\mathcal{A}_{X^2}) \xrightarrow{\sim} j^*(\mathcal{A}_X \boxtimes \mathcal{A}_X)$.

Goal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Example - lattice vertex algebra [Borcherds]

Let $(L, (\cdot,\cdot))$ be an even lattice.

Let $V_L = \mathbb{C}[L] \otimes \mathrm{Sym}(L(1) \oplus L(2) \oplus \cdots)$, with the natural bialgebra structure:

- Generators are denoted by $e^\alpha \in \mathbb{C}[L]$, $T^{(i)}(e^\alpha) \in L(i)$, ($\alpha$ in a basis of L).
- $\Delta(e^\alpha) = e^\alpha \otimes e^\alpha$; $\Delta(T) = T \otimes 1 + 1 \otimes T$.

Define $(V^L(I) = \bigotimes_I V^L) \in \mathrm{Fun}(Fin, A, T, S_B)$.

Now define a "bicharacter"

$$r : V_L \otimes V_L \to \mathbb{C}[(x-y)^{\pm 1}]$$

- $r(e^{\alpha} \boxtimes e^{\beta}) = \epsilon_{\alpha,\beta}(x-y)^{(\alpha,\beta)}$.
- $r(Tu \boxtimes v) = \frac{d}{dx}(r(u \boxtimes v))$, etc.

This allows us to "twist" the natural commutative multiplication on V^L to get a singular multiplication map:

$$\mu : V^L(1) \otimes V^L(2) \to V^L(1:2).$$

Indeed, we define

$$V_L \otimes V_L \to V_L \otimes V_L \otimes C[(x-y)^{\pm 1}]$$
$$u \boxtimes v \mapsto \sum u_{(1)} v_{(1)} r(u_{(2)} \boxtimes v_{(2)}).$$

Then $\theta_B(V^L)$ is the well-known lattice vertex algebra structure on the vector space V_L.

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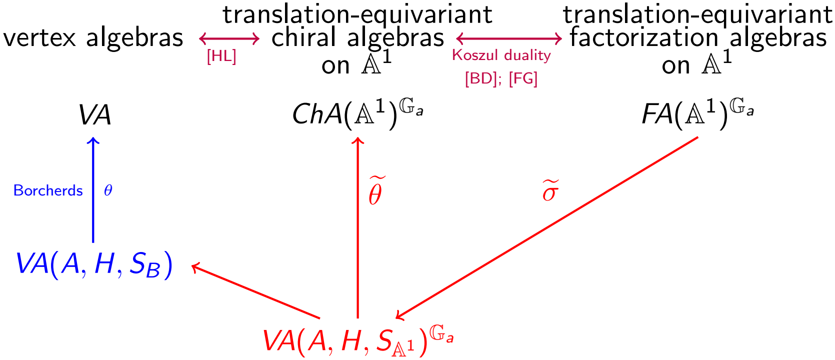

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These data are subject to a bunch of axioms.