Name: SOLUTIONS

netid.:

Math 402: Exam 3

Fall semester 2018

- Do not forget to write your name and netid on top of this page.
- No notes, books, calculators, or other exam aids are allowed. You may use a ruler and colored pens or pencils if you wish.
- Turn your cell phones off and put them away. **No use of cell phones** or other communication devices during the exam is allowed.
- Write your answers **clearly and fully** on the sheets provided. If you need additional paper, raise your hand.
- **Do not tear pages** off of this exam. Doing so will be considered cheating.
- The exam consists of 5 problems and 7 pages. Check that your exam is complete.
- You have **50 minutes** to complete the exam.

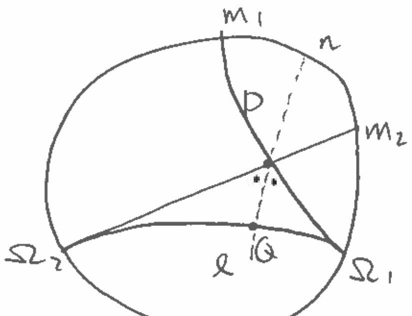
Good luck!!

Problem	1	2	3	4	5	$\sum$
Total possible	20	20	27	21	12	100
Your points						

20

Problem 1: (8 + 5 + 7 = ~~19~~ Points) Let ℓ be a line, and P a point not on ℓ.

1. Consider the two limiting parallels m_1 and m_2 to ℓ through P. Let n be a line which bisects the angle made by m_1 and m_2 at P and intersects ℓ at some point Q. Is it true that n is perpendicular to ℓ at Q? Why or why not?



Solution 1: Let n' be the perpendicular from P to ℓ. We know it makes congruent angles with m_1 and m_2 (angle of parallelism), so it is the line bisecting the angle at P made by m_1 and m_2. i.e. it is n.

Solution 2: By SA congruence, $PQ\Omega_1 \cong PQ\Omega_2$, and in particular $\angle \Omega_1 QP \cong \angle \Omega_2 QP = 90°$

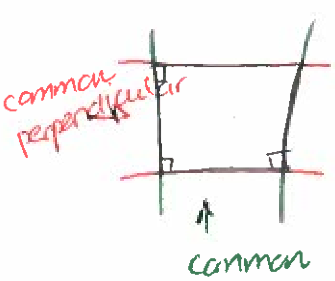
2. Define the *angle of parallelism* of ℓ at P.

It is the angle at P made by the limiting parallel (m_1 or m_2) and the perpendicular n to ℓ.

3. Can the lines ℓ and m_1 form opposite sides of a Lambert quadrilateral? Why or why not?

No. Opposite sides of a Lambert quadrilateral always have a common perpendicular

$\Rightarrow$ they are ultraparallel, not limiting parallel.



Problem 2: (16 + 4 = 20 Points)

1. Let F be a figure in the Euclidean plane such that the symmetry group Sym(F) has exactly five elements. For each of the following statements, say whether it is true or false, justifying your answers.

   (a) F has translational symmetry.

4 No. if $T_v \in Sym(F)$ is a translation then so is $T_v^n = T_{nv}$ $\forall n \geq 0$. $\Rightarrow Sym(F)$ is infinite #

   (b) Sym(F) has exactly as many rotations as it does reflections.

4 No. Sym(F) has no reflections, because if it did it would have even size.

   (c) Sym(F) contains only rotations.

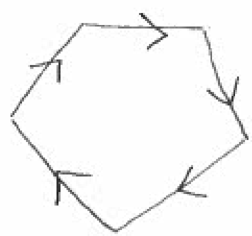
4 Yes. See (b).

   (d) Sym(F) might contain a rotation by 180°, but we can't tell for sure.

4 No. SymF is generated by Rot_{72} $''R$ ($72 = 360/5$) and contains only id, Rot_{72}, Rot_{144}, Rot_{216}, Rot_{288}

2. Draw a picture of a figure F whose symmetry group has exactly five elements.

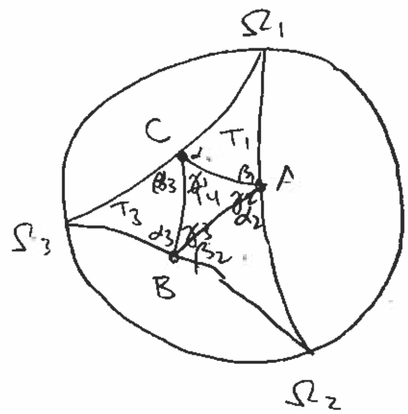
4



Problem 3: (14 + 13 = 27 Points)

(a) We did not prove, but it is true, that the area of an omega triangle $PQ\Omega$ is equal to $k^2(180 - (\angle PQ\Omega + \angle QP\Omega))$.

Now suppose that you are given three omega points, $\Omega_1, \Omega_2, \Omega_3$, which define a figure in the hyperbolic plane. Prove that this figure has area $k^2 180$.



$$\text{Area}(\Omega_1 \Omega_2 \Omega_3) = \sum_{i=1}^{4} \text{Area}(T_i)$$

By above fact

$$= \sum_{i=1}^{3} k^2(180-(\alpha_i+\beta_i))$$

Area(Δ) // By k^2 defect(Δ)

$$+ k^2(180-(\gamma_1+\gamma_2+\gamma_3))$$

$$= 4\cdot 180k^2 - k^2((\alpha_1+\gamma_1+\beta_3) + (\alpha_1+\gamma_2+\beta_1) + (\alpha_3+\gamma_3+\beta_2))$$

$$= 4\cdot 180k^2 - 3\cdot 180k^2$$

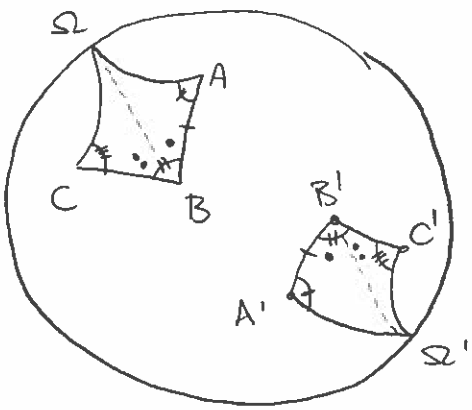
$$= 180k^2.$$

(b) An *omega-quadrilateral* is a figure in the hyperbolic plane with three real vertices and one omega-point as a vertex. Let $AB C\Omega$ and $A'B'C'\Omega'$ be two omega-quadrilaterals satisfying the following congruences:

$\angle CBA \cong \angle C'B'A'$

$\angle BA\Omega \cong \angle B'A'\Omega'$; $\quad \cancel{\angle CAB} \cong \angle C'A'B'$; $\quad \angle BC\Omega \cong \angle B'C'\Omega'$; $\quad \overline{AB} \cong \overline{A'B'}$.

Prove that $\overline{BC} \cong \overline{B'C'}$.



Consider the Ω-triangles $AB\Omega$ and $A'B'\Omega'$

By SA congruence, $AB\Omega \cong A'B'\Omega'$

$\Rightarrow \angle A'B'\Omega' \cong \angle AB\Omega$.

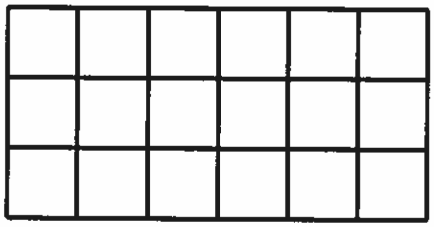
$\therefore \angle CB\Omega \cong \angle C'B'\Omega'$

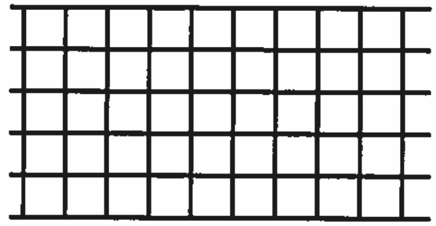
Now by AA congruence for Ω-triangles,

$BC\Omega \cong B'C'\Omega'$

and so $BC \cong B'C'$

Problem 4: (2 + 3 + 5 + 4 + 7 = 21 Points) In Euclidean geometry, we have three infinite families of regular tilings of type (n, k), where $n = 3, 4, 6$. (For each choice of n there are infinitely many tilings, because we can choose the tile to have any side length $\lambda > 0$. Here is a picture of two different tilings of type $(4, 4)$.)





In hyperbolic geometry, we can form a tiling of type $(5, 6)$, for example. Let P be the tile. Answer the following questions about P and about this tiling. (Give justification, wherever necessary. You should probably draw a picture.)

(a) How many sides does P have?

• $n = 5$

2

(b) What is the angle measure of an interior angle of P?

3 

• $\frac{360}{k} = \frac{360}{6} = 60$ because 6 corners meet at a vertex

(c) If we divide P into congruent isosceles triangles T with a common vertex at the centre of P, what are the angles (base and summit) in each of these isosceles triangles?

5 

$T =$ 

• $2\alpha = 60$, from (b) $\Rightarrow \alpha = 30°$

• $5\beta = 360 \Rightarrow \beta = 72°$

(d) What is the defect of P?

4 Defect $(P) = 180(5-2) - $ angle sum

$= 180(3) - 5 \cdot 60 = 540 - 300 = 240$

(e) Why is there a unique tiling of type $(5, 6)$, not an infinite family?

7 By AAA congruence, the side lengths of T is uniquely determined

and hence the side length of P

Problem 5: (6 + 2 + 4 =12 Points)

1. Let $z = x + iy$ be a non-zero complex number. Its inverse $\frac{1}{z}$ is also a complex number, so it can be expressed in the form $a + ib$. Find formulas for a and b.

METHOD 1: • $\frac{1}{x+iy} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \underset{\uparrow\, a}{\frac{x}{x^2+y^2}} + i \underset{\uparrow\, b.}{\frac{-y}{x^2+y^2}}$

METHOD 2: • $z \frac{1}{z} = 1 \Rightarrow (x+iy)(a+ib) = (ax-by) + i(ay+bx) = 1 + i0$

$ax-by = 1$ $\Rightarrow$ $\frac{-bx^2}{y} - by = 1$ $\Rightarrow$ $b = \frac{-y}{x^2+y^2}$,

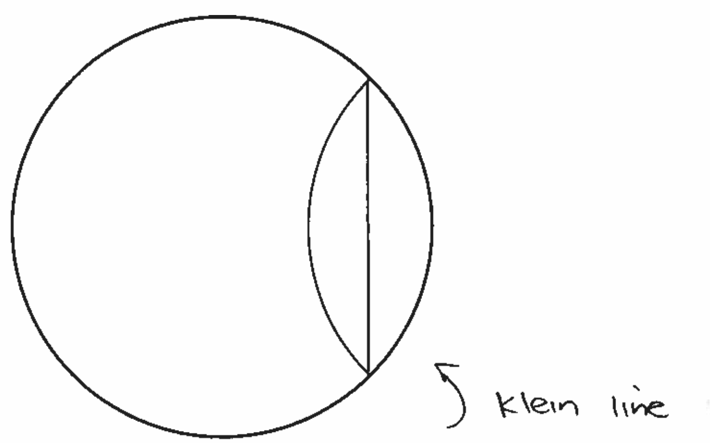
$ay+bx=0$

$a = \frac{x}{x^2+y^2}$.

$a \Rightarrow -\frac{bx}{y}$ ← assume $y \neq 0$, otherwise $\frac{1}{z} = \frac{1}{x}$ so $a = \frac{1}{x}$ and $b=0$

2. (a) The following shows a line in hyperbolic geometry. We have used two models this semester to study hyperbolic geometry, and we have given a map F between them. What is the name of the model this picture is using?

Poincaré



(b) Draw, on the same picture, what this line looks like in the *other* model of hyperbolic geometry (after applying F or F^{-1}).