MATH 402 Homework 1

Due Friday, September 7, 2018

Several of the exercises on this homework concern *groups*. We will make extensive use of groups later in the class. For now you can take groups to be an example of an axiomatic system.

Exercise 1.

a. *[5 points]* Write down the definition of a group (see page 24 of the book).

b. *[10 points]* Let G be the set of invertible functions from $\mathbb{R}$ to $\mathbb{R}$: an element of G is a function $f : \mathbb{R} \to \mathbb{R}$ such that there exists another function $g : \mathbb{R} \to \mathbb{R}$ such that $f \circ g = g \circ f = \mathrm{Id}_{\mathbb{R}}$ (that is: for any $x \in \mathbb{R}, f(g(x)) = g(f(x)) = x$). Consider the operation of composition of functions.

   Prove that this is a *model* for the axiomatic system that describes group structures. That is, prove that G is a group. For this you need to check the four axioms. In this case it will be convenient to check them in a different order than the way they are stated in the definition in the book (but of course it doesn't matter in the end which order you check them, as long as you know that they are all true).

   i. First show that composition of functions is associative: given three functions f, g, h, show that the compositions $f \circ (g \circ h)$ and $(f \circ g) \circ h$ are equal as functions on $\mathbb{R}$.
   ii. Next say which element should be the identity object of G, and prove that it satisfies the identity axiom.
   iii. Now show that each element has an inverse.
   iv. Finally, prove that A1 is satisfied: show that if f and f' are elements of G, then so is $f \circ f'$.

Exercise 2.

a. *[5 points]* Exercise 1.4.6: Let G be a group, and suppose x, y, z are elements of G such that $x \cdot z = y \cdot z$. Prove that $x = y$.

b. *[5 points]* Exercise 1.4.7: Let $x \in G$. Prove that $x \cdot e = e \cdot x$, using only the four axioms and the previous exercise.

c. *[5 points]* Exercise 1.4.8: prove that the identity element of a group G is unique. That is, suppose that G contains an element e' such that for every element $x \in G$, $x \cdot e' = x$. Show that $e' = e$, using only the four axioms and the previous two exercises.

Exercise 3. Consider the axiomatic system where the undefined terms consist of elements of a set S, and a set P consisting of certain pairs of elements of S. (So an element of P looks like (a, b), where a, b are in S, but not all pairs (a, b) are elements of P.) The axioms are as follows:

(1) If (a, b) is in P, then (b, a), is *not* in P.
(2) If (a, b) and (b, c) are in P, then (a, c) is in P.

a. Let $S_1 = \{1, 2, 3, 4\}$, and let $P_1 = \{(1, 2), (2, 3), (1, 3)\}$. Is this a model for the system? (Justify your answer—*always* justify your answers, unless specifically told otherwise.)

b. Let $S_2 = \mathbb{R}$, the set of all real numbers. Let $P_2 = \{(x, y) \mid x < y\}$. Is this a model for the system?

c. Use this to argue that the axiomatic system is not complete. In particular, can you add an axiom such that (S_1, P_1) is a model for the new system, but (S_2, P_2) is not?

Remember that in addition to the points assigned to each question, you will receive up to five further points for neatness and organization.