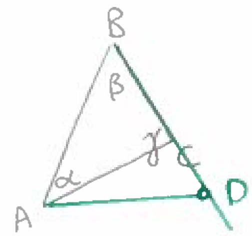
SOLUTIONS FOR HW 3 - Math 402, Sept. 2018.

Exercise 1: (a) Given a triangle ΔABC in which $AB > BC$, prove that the angles opposite these sides are also not congruent, and in fact that the angle opposite $\overline{AB}$ is larger than the angle opposite $\overline{BC}$.



- Choose a point D on ray $\overrightarrow{BC}$ s.t. $BD = AB$

  So ΔABD is an isosceles triangle, and the angles $\angle BAD \cong \angle BDA$.

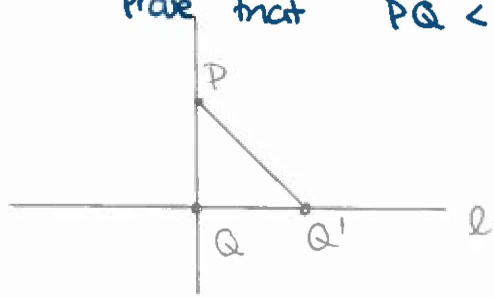
- Since $BC < BD$, we can see that C is <u>inside</u> the angle $\angle BAD$, and in particular, $\alpha < m(\angle BAD) = m(\angle BDA)$

- On the other hand, by the exterior angle theorem, applied to the triangle ΔACD and the exterior angle γ, $\gamma > m(\angle BDA)$.

$\therefore \alpha < \gamma$ as claimed.

(b) Let ℓ be a line and P a point not on ℓ.
Drop the perpendicular from P to ℓ, meeting ℓ at Q, say.
The <u>distance</u> from P to ℓ is PQ.

Let $Q' \in \ell$ be any point not equal to Q.
Prove that $PQ < PQ'$.



- if $PQ > PQ'$, the angles opposite would satisfy $\angle PQ'Q > \angle PQQ' = 90°$.
  But then the angle sum would be $> 180°$. #

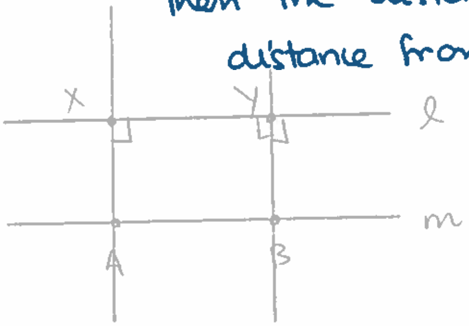
- also if $PQ = PQ'$, then we would have an isosceles triangle with both base angles $90°$, so angle sum is again $> 180°$ #

(Many other proofs are possible!)

(c) Prove that Playfair's Postulate implies the following:

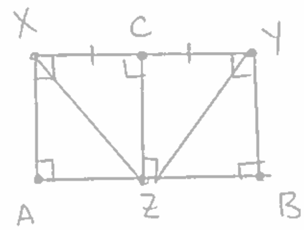
(*) Let ℓ, m be two parallel lines, and let $A, B \in$ ~~ℓ~~ m.

Then the distance from A to ℓ is equal to the distance from B to ℓ.



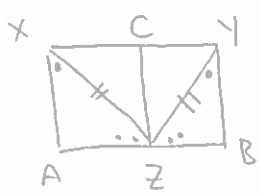
- First draw the perpendiculars to ℓ through A and B. Label the intersection points by X and Y.

- Let C be the midpoint of $\overline{XY}$, and drop a perpendicular from C to m.



- By Prop 29 (which follows from PP) we have right angles everywhere

$\Rightarrow$ By SAS, $\triangle XCZ \cong \triangle YCZ$.



- Since $\angle AXZ + \angle ZXC = 90°$
$\angle BYZ + \angle ZYC = 90°$
and $\angle ZXC \cong \angle ZYC$, we see that $\angle AXZ \cong \angle BYZ$

Similarly, $\angle AZX \cong \angle BZY$.

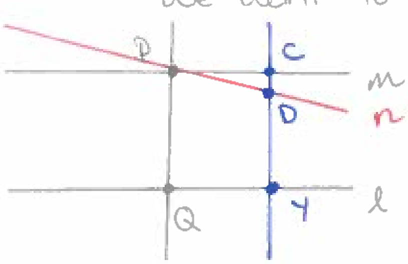
So by ASA, $\triangle AXZ \cong \triangle BYZ$, and in particular $AX = BY$ as claimed.

(d) prove the converse.

Let ℓ be a line, and P a point not on ℓ.

By drawing two perpendiculars we get a parallel line to ℓ through P.

We want to show it is unique.



Suppose towards a contradiction that n is another line through P parallel to ℓ.

Choose a point $Y \neq Q$ on ℓ, and draw a line t through Y perpendicular to ℓ, intersecting m and n at C and D.

By def, the distance from C to ℓ is CY; likewise the distance from D to ℓ is DY, but these must both be PQ. $\therefore C = D$ and $n = m$

Exercise 2:

(a) let c be a circle with centre O and radius r.
let P be any point. Define the power of P w.r.t. c.

Power of $P = (PO)^2 - r^2$.

(b) Prove that $\text{Power}(P) \begin{cases} >0 & \Rightarrow P \text{ is outside the circle} \\ =0 & \Rightarrow P \text{ is on the boundary} \\ <0 & \Rightarrow P \text{ is inside the circle.} \end{cases}$

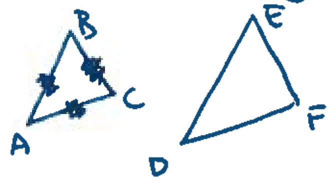
- $(PO)^2 - r^2 > 0 \Rightarrow PO > r$ (since both are positive)
  $\Rightarrow P$ is outside c.

- Likewise, $(PO)^2 - r^2 = 0 \Rightarrow PO = r$
  $\Rightarrow P$ is on the circle

  $(PO)^2 - r^2 < 0 \Rightarrow PO < r$
  $\Rightarrow P$ is inside the circle

Exercise 4. Prove SSS similarity:

Suppose we have



s.t. $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$.

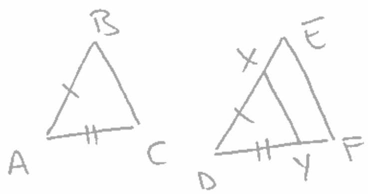
Prove that the triangles are similar.

- if $AB = DE$, then all sides are congruent, and $\triangle ABC \cong \triangle DEF$ by SSS congruence

So WLOG assume $AB < DE$; and hence $AC < DF$

Choose X on $\overline{DE}$ s.t. $DX = AB$

Y on $\overline{DF}$ s.t. $DY = AC$



- Now by SAS similarity $\triangle DXY \sim \triangle DEF$

In particular, $\frac{XY}{EF} = \frac{DX}{DE} = \frac{AB}{DE} = \frac{BC}{DF}$

$\Rightarrow XY = BC$.

$\therefore$ By SSS congruence, $\triangle ABC \cong \triangle DXY$, and since we already know that $\triangle DXY \sim \triangle DEF$, we conclude that $\triangle ABC \sim \triangle DEF$ as desired.