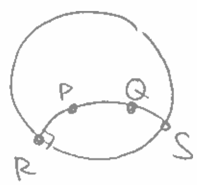
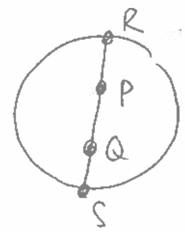
MATH 402 - HW5 Solutions.

Exercise 1. $d_p(P,Q) = \left| \ln\left(\frac{PS}{PR} \cdot \frac{QR}{QS} \right) \right|$

(a) draw a picture showing R and S.





(R and S can be swapped)

(b) show $d_p(P,Q) = 0 \Leftrightarrow P = Q$. (hint: compare the fractions $\frac{PS}{QS}$, $\frac{PR}{QR}$. what happens if $PS < QS$?

" $\Leftarrow$ " • if $P = Q$, $\frac{PS}{PR} \cdot \frac{QR}{QS} = 1$

$\Rightarrow d_p(P,Q) = |\ln(1)| = |0| = 0$.

" $\Rightarrow$ " if $d_p(P,Q) = 0$, $\ln\left(\frac{PS}{PR} \cdot \frac{QR}{QS}\right) = 0$

$\Rightarrow \frac{PS}{PR} \cdot \frac{QR}{QS} = 1$

$\Rightarrow \frac{PS}{QS} = \frac{PR}{QR}$.

* can also use SAS congruence

if $P \neq Q$, then $PS \neq QS$.

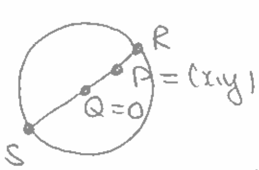
WLOG assume $PS < QS$.

then $PR > QR$, by inspection.

But then $\frac{PS}{QS} < 1$, $\frac{PR}{QR} > 1$ #.

So $P = Q$.

(c) if $Q = O$ = centre of unit circle, ~~show~~ simplify the formula.



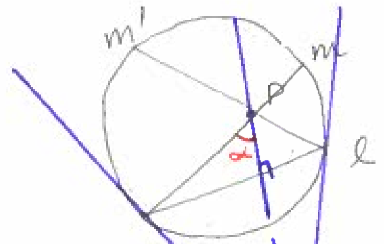
• $QR = QS = 1$

• if $PS = a$, $PR = 2 - a$

$\therefore d_p(P,Q) = \left| \ln\left(\frac{a}{2-a} \right) \right|$

Exercise 2:

(a) Draw the Klein disk, with a line ℓ, a point P not on ℓ, and two limiting parallels m, m' to ℓ at P.

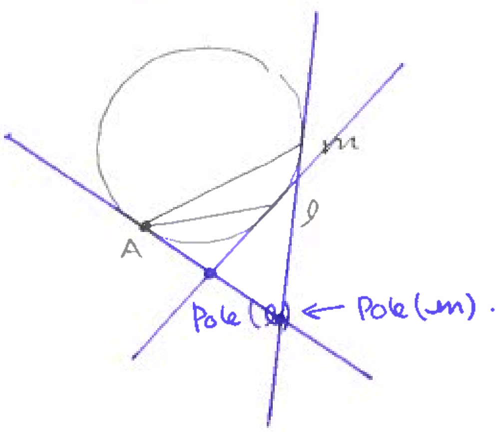


pole of ℓ.

(b) Draw the perpendicular line from P to ℓ. Label the angle of parallelism.

α = angle of parallelism.

(c) Prove that there is no Klein line perpendicular to ℓ and m.



A Klein line perpendicular to both ℓ and m must extend to a Euclidean line which passes through both Pole (ℓ) and Pole (m)

• But this Euclidean line is the tangent at A, which does not intersect the interior of the disk, so it does not correspond to a Klein line

* can also use: if n is perpendicular to ℓ at Q and m at P, then m is limiting parallel to ℓ at P with angle of parallelism $= 90°$ #

Exercise 3:

(a) Prove that a bijective function must have a unique inverse.

Let $f: S \to S$ be a bijective function.

Given any $s \in S$, surjectivity of f implies $\exists t \in S$ s.t. $f(t) = s$.

Injectivity of f implies t is unique.

Define $g(s) := t$; we claim $g = f^{-1}$

① $f \circ g = id$:

- take $s \in S$ as above, t s.t. $f(t) = s$

so $g(s) = t$.

$f \circ g(s) = f(t) = s \quad \forall s \in S$.

② $g \circ f = id$:

- take $x \in S$, and let $s = f(x)$.

$g \circ f(x) = g(s) =$ unique element t s.t. $f(t) = s$

$f(x) = s \Rightarrow$ by uniqueness $t = x$.

$\Rightarrow g \circ f(x) = x \quad \forall x \in S$.

Uniqueness of g: if f has another inverse, g', then

$f \circ g = f \circ g' \Rightarrow \forall s \in S,\ f(g(s)) = f(g'(s))$

$\Rightarrow$ (since f is injective) $g(s) = g'(s)$

$\Rightarrow g = g'$

(b) if f, g are invertible, check that $h = f \circ g$ is invertible with $h^{-1} = g^{-1} \circ f^{-1}$.

- $h \circ (g^{-1} \circ f^{-1}) = id$:

$\forall s \in S,\ h \circ (g^{-1} \circ f^{-1})(s) = f(g(g^{-1}(f^{-1}(s)))) = f(f^{-1}(s)) = s$.

- $(g^{-1} \circ f^{-1}) \circ h(s) = id$:

$\forall s \in S,\ (g^{-1} \circ f^{-1}) \circ h(s) = g^{-1}(f^{-1}(f(g(s)))) = g^{-1}(g(s)) = s$.

(c) Prove that if f is an isometry, then f^{-1} is also an isometry.

- we need to show that $\forall A, B$, $f^{-1}(A)f^{-1}(B) = AB$.

- Applying the isometry f to $X = f^{-1}(A)$, $Y = f^{-1}(B)$

$$f(X)f(Y) = XY$$

But $f(X) = f(f^{-1}(A)) = A$, $\quad f(Y) = f(f^{-1}(B)) = B$.

so this gives $\quad AB = f^{-1}(A)f^{-1}(B)$ as required.

(d) Prove that if f and g are isometries, $f \circ g$ is an isometry.

- $f : \mathbb{R}^2 \to \mathbb{R}^2$, $\quad g : \mathbb{R}^2 \to \mathbb{R}^2$

$$\Rightarrow f \circ g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

- we need to show that $\forall A, B$. $((f \circ g)A)((f \circ g)B) = AB$.

$$(f \circ g(A))(f \circ g(B)) = f(g(A))f(g(B))$$

$= g(A)g(B)$ since f is an isometry

$= AB$ since g is an isometry.

(e) Prove that the set of isometries forms a group.

- By part (d), composition gives a map

$\circ : G \times G \longrightarrow G$ $\quad$ (where $G = \{f : \mathbb{R}^2 \to \mathbb{R}^2 \mid f \text{ is an isometry}\}$)

- It is easy to see that $id \in G$,

and $f \circ id = f$, $id \circ f = f \quad \forall f \in G$.

so G has a unit

- associativity follows from associativity of composition.

- By part (c), $f \in G \Rightarrow f$ has an inverse $f^{-1} \in G$.