Math 402 - HW7.

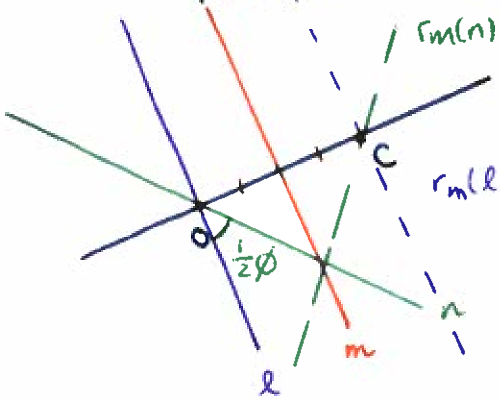
Exercise 1: Let $T \neq id$ be a translation.
Prove that the only invariant lines of T are those parallel to the translation vector v.

Let ℓ be a line.
Choose a representation $\vec{AB}$ of $\vec{v}$ where $A, B \notin \ell$; $B = T(A)$.
Take $P \in \ell$. We know $APT(P)B$ is a parallelogram (or a single line segment), so $\overleftrightarrow{AB} \parallel \overleftrightarrow{PT(P)}$.

"$\Rightarrow$" assume ℓ is invariant
then $T(P) \in \ell$, so $\overleftrightarrow{PT(P)} = \ell$,
so ℓ is parallel to $\overleftrightarrow{AB}$ and hence to $\vec{v}$.

"$\Leftarrow$" assume ℓ is parallel to $\vec{v}$ and hence to $\overleftrightarrow{AB}$.
- Since $\overleftrightarrow{PT(P)}$ is also parallel to $\overleftrightarrow{AB}$ and passes through P,
Playfair's Postulate implies $\overleftrightarrow{PT(P)} = \ell$
so $T(P) \in \ell$.
since $P \in \ell$ was arbitrary, we see that $T(\ell) \subset \ell$. $\square$.

Exercise ~~2~~ 3: (a) Suppose we are given a coordinate system with origin O.
~~Let~~ Rot_ϕ; let $C = (x, y) \neq O$ and let $T(a,b) = (a,b) + (x,y)$.
Prove that $T \circ Rot_\phi \circ T^{-1}$ is a rotation about C by ϕ.



Let ℓ be the line through O perpendicular to $\overleftrightarrow{OC}$.

Let m be the perpendicular bisector of $\vec{OC}$

Let n be the angle bisector of the angle made by ℓ and $Rot_\phi(\ell)$.

So $T = r_m \circ r_\ell$, $T^{-1} = r_\ell \circ r_m$, $Rot_\phi = r_n \circ r_\ell$ (i)

So $T \circ Rot_\phi \circ T^{-1} = (r_m \circ r_\ell) \circ (r_n \circ r_\ell) \circ (r_\ell \circ r_m)$

$= r_m \circ r_\ell \circ r_n \circ r_m$ (ii)

$= r_m \circ r_\ell \circ r_m \circ r_m \circ r_n \circ r_m$

$= r_{r_m(\ell)} \circ r_{r_m(n)}$

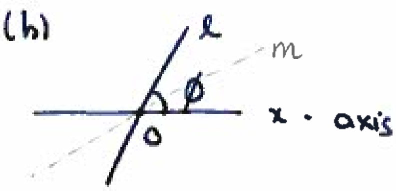
ℓ and n intersect at O with angle $\phi/2$

$\Rightarrow r_m(\ell)$ and $r_m(n)$ intersect at $r_m(O) = C$ with angle $\phi/2$ (iii)

$\Rightarrow r_m(\ell) \circ r_m(n) = Rot_{C, 2(\phi/2)} = Rot_{C,\phi}$

(Note that the orientation matches – it's not $-\phi$)

(b)



Claim: $r_\ell = Rot_\phi \circ r_x \circ Rot_{-\phi}$

Let m be the angle bisector of the angle made by x-axis and ℓ. So $Rot_\phi = r_m \circ r_x$.

$\Rightarrow Rot_\phi \circ r_x \circ Rot_{-\phi} = (r_m \circ r_x) \circ r_x \circ (r_x \circ r_m)$

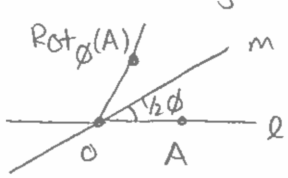
$= r_m \circ r_x \circ r_m$

$= r_n$. where $n = r_m(x\text{-axis})$
$= \ell$.

Exercise 2:

(a) prove that $(Rot_\phi)^{-1} = Rot_{-\phi}$, using reflections.

Choose a point $A \neq O$; let $\ell = \overleftrightarrow{OA}$ and let m be the angle bisector of $\angle A O Rot_\phi(A)$



So $Rot_\phi = r_m \circ r_\ell$

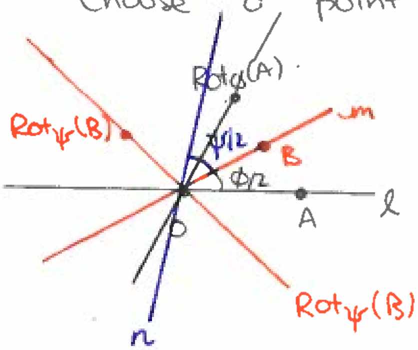
$\Rightarrow (Rot_\phi)^{-1} = r_\ell \circ r_m$, which is rotation about $O = \ell \cap m$ by angle $2(\text{angle from } m \text{ to } \ell)$

$= 2(-\frac{1}{2}\phi) = -\phi$.

$\Rightarrow (Rot_\phi)^{-1} = Rot_{-\phi}$.

(b). Prove that $Rot_\psi \circ Rot_\phi$ is again a rotation using reflections.

Choose a point $A \neq O$ and let $\ell = OA$.



let m be the angle bisector of $\angle AORot_\phi(A)$

let B be a point on m not equal to O.

let n be the angle bisector of $\angle BORot_\psi(B)$.

$\Rightarrow Rot_\phi = r_m \circ r_\ell$, and m and ℓ form angle $1/2\, \phi$

$Rot_\psi = r_n \circ r_m$ and n and m form angle $1/2\, \psi$.

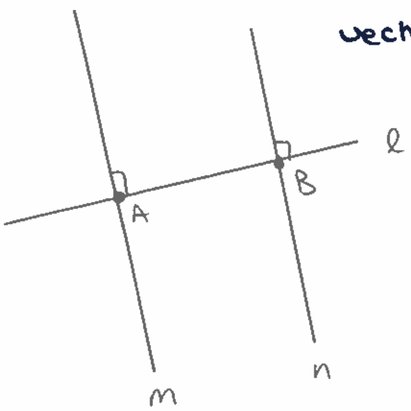
$\Rightarrow Rot_\psi \circ Rot_\phi = r_n \circ r_m \circ r_m \circ r_\ell = r_n \circ r_\ell$.

and n and ℓ meet at O forming angle $1/2\,(\phi+\psi)$

$\Rightarrow Rot_\psi \circ Rot_\phi = Rot_{\phi+\psi}$. a rotation.

(c) Let A and B be two different points. Let $R_1 = Rot_{A,180}$, $R_2 = Rot_{B,180}$.

Prove that $R_2 \circ R_1$ is a translation, and identify the displacement vector.



let $\ell = \overleftrightarrow{AB}$

let m be the perpendicular bisector to ℓ at A, and n the perpendicular bisector to ℓ at B

$\Rightarrow R_1 = r_\ell \circ r_m$, $R_2 = r_n \circ r_\ell$.

$\Rightarrow R_2 \circ R_1 = r_n \circ r_\ell \circ r_\ell \circ r_m = r_n \circ r_m$,

and n and m are parallel.

$\Rightarrow r_n \circ r_m$ is translation, with displacement vector $2\overrightarrow{AB}$.

(d) $\mathcal{R} = \{\text{rotations}\}$ $\mathcal{R}_o = \{\text{rotations about } o\}$

Is $\mathcal{R}$ a group? Is $\mathcal{R}_o$ a group?

- $\mathcal{R}$ is not a group: by (c) it is not closed under composition.

- $\mathcal{R}_o$ is a group: by (b) it is closed under composition
  by (a) it is closed under inverses.

Exercise 4: $G = T_{AB} \circ r_\ell$, $\vec{AB} \neq \vec{0}$, $\overleftrightarrow{AB} \parallel \ell$.

Show that

(a) the only invariant line under G is ℓ.

- ℓ is clearly invariant (fixed under r_ℓ, invariant under T_{AB}).
- suppose m is an invariant line, so $T_{AB} \circ r_\ell(m) = m$

$\Rightarrow r_\ell(m) = T_{AB}^{-1}(m)$.

  - translation preserves direction of lines, so $T_{AB}^{-1}(m) \parallel m$
  - but the only way $r_\ell(m) \parallel m$ is if $m \parallel \ell$, $m = \ell$, $m \perp \ell$.

- if $m \parallel \ell$, $T_{AB}^{-1}(m) = m$, but $r_\ell(m) \neq m$. #
- if $m \perp \ell$, $T_{AB}^{-1}(m) \neq m$, but $r_\ell(m) = m$ #

So the only option is $m = \ell$.

(b) G has no fixed points

Note that $G^2 = (T_{AB} \circ r_\ell) \circ (r_\ell \circ T_{AB})$
$= T_{2AB}$

if $G(P) = P$, $G^2(P) = P$, so P is also a fixed point of T_{2AB}

But $\vec{AB} \neq \vec{0} \Rightarrow 2\vec{AB} \neq 0 \Rightarrow T_{2AB}$ has no fixed points

$\therefore G$ has no fixed points.

Alternate approach: use coordinates (with ℓ = x-axis)

$G: (x,y) \mapsto (x+a, -y)$.