MATH 402 Homework 11

Optional — highly recommended for exam preparation!
Come to office hours or make an appointment to discuss your solutions.

Exercise 1. Do the attached exercise on equivalence and defect. (Sorry I didn't type it!—see the webpage for the link.)

Exercise 2. Practice with complex numbers.
⋆ *especially important if you haven't seen complex numbers much before* ⋆

a. The *exponential* of a complex number $z = x + iy$ is defined by

$$e^z = e^x(\cos y + i \sin y).$$

In particular, we have that $e^{iy} = \cos y + i \sin y$ (this is called *Euler's formula*).

Prove that $e^{i\phi}e^{i\psi} = e^{i(\phi+\psi)}$. (Be careful that you're only using facts you know about *complex* exponentials and multiplication of complex numbers, not making assumptions based on how you know things work for exponentials of real numbers.)

b. Rewrite each of the following expressions in the form $a + ib$ for a, b real numbers:

$$\frac{1}{2i}; \quad \frac{1+i}{1-i}; \quad \frac{1}{2+4i}.$$

(Hint: *rationalize* the denominator: multiply both the numerator and the denominator by the complex conjugate of the denominator.)

Exercise 3. Practice with stereographic projection. Suppose that $z = x + iy$ is a point of the complex plane corresponding to the point $P = (X, Y, Z)$ of the unit sphere under stereographic projection. Review the proof that $x = \frac{1}{1-Z}X$ and $y = \frac{1}{1-Z}Y$

a. Conversely, show that

$$X = \frac{2x}{x^2+y^2+1}, \quad Y = \frac{2y}{x^2+y^2+1}, \quad \frac{x^2+y^2-1}{x^2+y^2+1}.$$

(Hint: recall that the line ℓ between two points A and B has the form $\{tA + (1-t)B \mid t \in \mathbb{R}\}$.)