MATH 402 Review for October 22–26

Topics: Matrix forms for isometries; compositions of isometries; symmetries. (5.5, 5.6, 5.7, 5.8)
These were covered on Worksheet 5 and in lecture. This material will also appear in Homework 8.

1. **Recall from last week:**

# of reflections	Relations between the lines	Name of isometry	Fixed point set
0		identity	everything
1: r_ℓ		reflection	ℓ
2: $r_\ell \circ r_m$	$\ell = m$	identity	everything
2: $r_\ell \circ r_m$	ℓ and m are parallel	(non-identity) translation	$\varnothing$
2: $r_\ell \circ r_m$	$\ell \cap m = \{O\}$	(non-identity) rotation	$\{O\}$
3: $r_\ell \circ r_m \circ r_n$	exactly two of the lines intersect at a single point P	glide reflection	$\varnothing$
3: $r_\ell \circ r_m \circ r_n$	any situation other than the previous line	reflection	a single line (we have to work to see which one)

2. **Matrix forms for isometries:**

   (a) Make sure you are comfortable with multiplying 2×2 and 3×3 matrices.

   (b) We saw that we needed to work with the *affine model* for the Euclidean plane, in which the point (x, y) is represented by the column vector $(x, y, 1)$. The reason for this is that translations cannot be represented by multiplication by a 2×2 matrix, but we can make it work out with 3×3 matrices.

   (c) We have the following:

$$T_{(v_1,v_2)} = \begin{bmatrix} 1 & 0 & v_1 \\ 0 & 1 & v_2 \\ 0 & 0 & 1 \end{bmatrix}; \qquad R_{0,\phi} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$r_{y\text{-axis}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \qquad r_{x\text{-axis}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3. These give us matrix forms of certain simple isometries, but we can multiply them together (corresponding to composing the isometries) to get more general matrix forms. Here are some examples.

   - On HW 7, you wrote a formula for rotation around a point C in terms of rotation around 0 by using the translation T that takes O to C.
   - You also wrote a formula for reflection across a line ℓ that passes through 0 in terms of reflection across the x-axis, and the rotation R that takes the x-axis to ℓ.

4. **Composing isometries**

   (a) We proved the following (not all this week):

   - The composition of two translations is a translation (just add the displacement vectors).

- The composition of a rotation with a rotation is a translation if the angles sum to 0 mod 360 (example to keep in mind: two half-turns), and it's a rotation if the angles sum to anything else (example to keep in mind: two rotations about the same point give a rotation about that same point, and it's only a translation if it's the identity, which only happens when the angles sum to 0 or 360).
- The composition of a (non-identity) rotation R with a translation is a rotation (note: it's either a rotation or a translation; if it were another translation, then the rotation would be equal to the composition of two translations, which would be a translation, which is a contradiction). The angle of rotation is the same as that of R.
- Conjugation of a reflection by any isometry gives another reflection: $f \circ r_m \circ f^{-1} = r_{f(m)}$.

(b) Techniques we used in the proofs:

- Choose a nice coordinate system so that you can write down the matrices for the isometries you want to compose. Multiply them. See what you get.
- Remember that once you know that an isometry fixes a line ℓ, it is either r_ℓ or the identity.
- We know that an isometry is orientation-preserving if and only if it can be written as a composition of an even number of reflections, and it is orientation-reversing if and only if it can be written as a composition of an odd number of reflections. So if we count up the number of reflections in the composition, we can see straight away whether it's even (and hence a rotation or translation) or odd (and hence a reflection or glide reflection).
- If you're composing two isometries f, g which are each made up of more than one reflection, remember that you have some choice in choosing the way to write each of them as a composition. Try to choose lines of reflection such that you get cancellations when you compose f and g.
- If you can't arrange direct cancellations (e.g. by having two of the same reflection right next to each other), remember that if you have the same reflection r_m sandwiching another reflection r_ℓ, we can simplify the expression:

$$r_m \circ r_\ell \circ r_m = r_{r_m(\ell)}.$$

  Notice that it is sometimes helpful to stick in extra r_m's (always in pairs! $r_m \circ r_m$) to get cancellations. See the practice question for an example of how this can help.

Practice Question

On the homework, you proved that $f \circ r_m \circ f^{-1} = r_{f(m)}$. Generalize this result: what if we replace r_m by a rotation, translation, or glide reflection? Write this isometry g as a composition (e.g. $r_1 \circ r_2 \circ r_3$ of reflections). Notice that $f \circ r_1 \circ r_2 \circ r_3 \circ f^{-1} = f \circ \circ r_1 f^{-1} \circ f \circ r_2 \circ f^{-1} \circ f \circ r_3 \circ f^{-1}$.