MATH 402 Review for October 29–31 and November 6–8

Topics: Finite plane symmetry groups (6.1); limiting parallels (7.3).
These were covered in lecture. This material will also appear in Homework 9.

1. **Recall from last week:** Let F be a figure in the plane. We set $\mathrm{Sym}(F) = \{f$ an isometry such that $f(F) = F\}$. This forms a group.

2. **Finite plane symmetry groups**

   (a) The symmetry group $\mathrm{Sym}(F)$ is called *finite* if it contains only finitely many elements.

   (b) The *order* of an element g of a group G is the smallest positive integer $o(g)$ such that $g^n = G$. If there is no such integer, we say $o(g) = \infty$. In a finite group, all elements have finite order.

   - $o(\mathrm{id}) = 1$, $o(r_\ell) = 2$, $o(T_v) = \infty$, $o(G_{\ell,AB}) = \infty$, and $o(R_{O,\phi})$ depends on ϕ.
   - It follows that finite symmetry groups can't have translations or glide reflections, only reflections and rotations (and the identity, which is a special rotation).

   (c) Let G be a finite plane symmetry group. Then

   - All rotations in G have the same center of rotation, call it O.
   - The set of all rotations in G forms a subgroup, and it is cyclic, generated by the rotation $R = R_{O,\alpha}$ with $\alpha > 0$ minimal. (*Cyclic* means all elements in this subgroup are of the form R^j for some j.)
   - If G has a reflection r_ℓ, then $O \in \ell$.
   - If G has exactly n rotations (including the identity), $\{R^0 = \mathrm{id}, R, R^2, \ldots, R^{n-1}\}$, and in addition if G has a reflection r_ℓ, then in fact G has exactly n reflections, all of the form $r_\ell \circ R^j, (j = 0, \ldots, n-1)$.

   (d) It follows that every finite plane symmetry group has some number n of rotations, and then either it has no reflections (in this case the group is cyclic of order n), or it has exactly n reflections (and then we call it *dihedral* of order $2n$, and write D_n). This is called **Leonardo's theorem**.

3. **Limiting parallels in hyperbolic geometry (without a model!)**

   (a) Definition: Let ℓ be a hyperbolic line and P a point not on ℓ. A line m through P is *limiting parallel* to ℓ through P if

   i. m is parallel to ℓ; and
   ii. any ray $\overrightarrow{PX}$ interior to the angle made by m and ℓ intersects ℓ.

   (b) The **Fundamental theorem of parallels in hyperbolic geometry** tells us that given any line ℓ and any point P not on ℓ there are exactly two *limiting parallels* to ℓ through P, and that these limiting parallels form congruent acute angles with the perpendicular from P to ℓ, called the *angle of parallelism.*

   (c) All other parallel lines through P are called *ultraparallel* to ℓ.

   (d) If m is limiting parallel to ℓ through P, and P' is another point on m, then m is also limiting parallel to ℓ through P'. This means we can just say "m is limiting parallel to ℓ" to mean that for any choice of point P on m, m is limiting parallel to ℓ at P.

   (e) If m is limiting parallel to ℓ then ℓ is limiting parallel to m.

Practice Questions

1. In lecture, we studied the symmetry group of an equilateral triangle, and showed that it contains exactly six elements, one for every permutation of $\{1, 2, 3\}$.

   Think about the symmetry group of a square. Show that every symmetry f of the square gives a permutation of $\{1, 2, 3, 4\}$, and show that two different symmetries f, g, must give different permutations. But does every permutation correspond to some symmetry of the square?

2. Draw pictures in the Klein model and the Poincaré model to explain why facts 3(d) and 3(e) from the previous page are true in these models.