MATH 595 Thursday 8 March

Dualizing sheaves; higher direct image

(1) **Chapter III, Exercise 7.3.**

Let $X = \mathbb{P}^n_k$. Recall that $\Omega^p_X = \bigwedge^p \Omega_X$, the sheaf of differential p=forms on X. Prove that for any integers $0 \leq p, q \leq n$, we have

$$H^q(X, \Omega^p_X) = \begin{cases} 0, & p \neq q; \\ k, & p = q. \end{cases}$$

Hints: Use the following facts:

- Recall from the section on differentials that for $X = \mathbb{P}^n$ we have a short exact sequence expressing Ω_X in terms of the sheaves $\mathcal{O}_X$ and $\mathcal{O}_X(-1)$.
- Also recall (from II.5) that whenever we have a short exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$$

of locally free sheaves of finite rank, for any $r > 0$ we have a finite filtration of $\bigwedge^r \mathscr{F}$

$$\bigwedge^r \mathscr{F} = F^0 \supseteq F^1 \supseteq F^2 \supseteq \dots$$

with the property that for any $p > 0$, the quotient F^p/F^{p+1} is isomorphic to $\bigwedge^p \mathscr{F}' \otimes \bigwedge^{r-p} \mathscr{F}''$.

(2) **Chapter III, Exercise 8.3.** Let $f : X \to Y$ be a morphism of ringed spaces, $\mathscr{F}$ an $\mathcal{O}_X$-module, and $\mathscr{E}$ a locally free $\mathcal{O}_X$-module of finite rank. Prove the following generalization of the projection formula:

$$R^i f_*(\mathscr{F} \otimes f^*\mathscr{E}) \cong R^i f_*(\mathscr{F}) \otimes \mathscr{E}.$$