MATH 595 Tuesday April 10

More practice with Hurwitz's Theorem

(1) **IV.2.3 (b–e)** Let $X \subset \mathbb{P}^2$ be a curve of degree d.

Recall the *dual projective plane* $(\mathbb{P}^2)^*$: it is the set of lines in $\mathbb{P}^2$, which we can identify with $\mathbb{P}^2$ by associating the line $L = \{a_0x + a_1y + a_2z = 0\}$ with the point $(a_0 : a_1 : a_2)$. We define a map

$$X \to (\mathbb{P}^2)^*$$

by sending a point $P \in X$ to the tangent line T_PX. The image of this map is a curve X^* called the *dual curve.*

We also need the following terminology: a line L in $\mathbb{P}^2$ is a multiple tangent of X if it is tangent to X at more than one point. It is an inflectional tangent of X if it is tangent to X at an inflection point.

(a) Thinking about the geometry of this situation, convince yourself of the following (a rigorous proof is not necessary): if L is a multiple tangent of X, tangent to X at the points $P_1, \ldots, P_r$, and if none of the P_i is an inflection point, show that the corresponding point of X^* is an ordinary r-fold point.
Conclude that X has only finitely many multiple tangents.

(b) Let $O \in \mathbb{P}^2$ be a point which is not on X, nor on any inflectional or multiple tangent of X, and let L be a line not containing O. Then we can define a morphism $\psi : X \to L$ by projection from O.
Prove that ψ is ramified at $P \in X$ if and only if OP is tangent to X at P. Furthermore, in that case, the ramification index is 2.
(Hint: choose coordinates so that $O = (0, 0) \in \mathbb{A}^2 = \{z \neq 0\}$, $P = (0, 1)$, and $L = \{z = 0\}$ is the line at infinity. Write the formulas for ψ and $\psi^\#$.)

(c) Now use Hurwitz's theorem to conclude that there are exactly $d(d-1)$ tangents of X passing through O, and thus that the degree of the curve X^* is $d(d-1)$.

(d) Show that for all but finitely many points in X, a point $O \in X$ lies on exactly $(d+1)(d-2)$ tangents of X, not counting the tangent at O itself.
(Hint: show that for suitable $O \in X$, the results of (b) can be extended, giving a map $\psi : X \to L$ of degree $d-1$ with properties as described in (b).) Now apply Hurwitz's theorem to ψ.)

(e) Show that the degree of the morphism ϕ from **IV.2.3 (a)** (which was on last Thursday's worksheet) is $d(d-1)$.

(f) Use Hurwitz's theorem together with your results from above to prove that for $d \geq 2$ X has $3d(d-2)$ inflection points (counted appropriately).