MATH 595 Thursday April 12

Embeddings of curves in $\mathbb{P}^n$

(1) Let X be a curve in $\mathbb{P}^n$, and let O be a point in $\mathbb{P}^n \setminus X$. Let $\phi : X \to \mathbb{P}^{n-1}$ be the morphism defined by projection from the point O. This morphism corresponds to a linear system $\mathfrak{d}$. Prove that $\mathfrak{d} = \{X.H \mid H \text{ is a hyperplane in } \mathbb{P}^n \text{ containing } O\}$.

Hint: Choose projective coordinates for $\mathbb{P}^n$ and O so that you can write down ϕ explicitly. Remember that $\mathfrak{d}$ is the linear system consisting of the divisors of zeroes of sections of $\mathcal{O}_X(1) \simeq \phi^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ corresponding to the generators of $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$.

(2) **Exercise IV.3.2**

Let X be a plane curve of degree d.

(a) Show that the effective canonical divisors on X are the divisors $X.L$, where L is a line in $\mathbb{P}^2$.
Hint: First show that $\{X.L\} \subset |K|$. Now show that they are projective spaces of the same dimension.

(b) Let D be any effective divisor of degree 2 on X. Prove that $\dim |D| = 0$.
(Hint: find a line L such that $X.L = D + D'$. Use the above result together with the fact that K is very ample.)

(c) Conclude that X is not hyperelliptic (i.e. does not have a degree 2 map to $\mathbb{P}^1$).

(3) **Exercise IV.3.1** Let X be a curve of genus 2. Show that D is very ample if and only if the degree of D is at least five.

(Hint: consider divisors of degree 4 first, and show they can't be very ample. What about divisors of degree 3 ...?)

(4) **Exercise IV.3.3** Let X be a curve of genus $g \geq 2$, and assume that X is a complete intersection in $\mathbb{P}^n$. Prove that the canonical divisor K is very ample. Use the previous question to conclude that a curve of genus 2 is not a complete intersection in any $\mathbb{P}^n$.

(Hint: Use Ex. II.8.4 to write down a (pretty) explicit representative of K.)