MATH 595 Thursday 19 April

Riemann–Roch for surfaces; the adjunction formula

(1) **Exercise V.1.2**

Let H be a very ample divisor on a surface X corresponding to an embedding $X \subset \mathbb{P}^n$. We saw that we can assume H is an irreducible, non-singular curve on X. Let g_H denote the genus of this curve.

If we write the Hilbert polynomial of X as

$$P(z) = \frac{1}{2}az^2 + bz + c,$$

prove that $c = 1 + p_a$, $a = H^2$, and $b = \frac{1}{2}H^2 + 1 - g_H$. (Hint: use Riemann–Roch for a and c, and use the adjunction formula to recover b.)

Conclude that the degree of X in $\mathbb{P}^n$ is exactly H^2; furthermore, if $C \subset X$ is any curve, the degree of C in $\mathbb{P}^n$ is $C.H$.

(2) **Exercise V.1.3(a)**

Let D be any effective divisor on X. Use Riemann–Roch to extend the adjunction formula to any such D (even if it's singular, reducible, etc.):

$$D.(D + K) = 2p_a(D) - 2.$$

(Recall that $p_a(D) = 1 - \chi(\mathcal{O}_D)$, for D any projective scheme of dimension 1.)

(3) **Exercise V.1.4, V.1.5**

(a) Let X be a surface of degree d in $\mathbb{P}^3$. Suppose that X contains a straight line $C = \mathbb{P}^1$. Prove that $C^2 = 2 - d$.
(Hint: use the adjunction formula and solve for C^2. You will need to think about what K_X looks like.)

(b) If X is again a surface of degree d in $\mathbb{P}^3$, show that $K^2 = d(d - 4)^2$.
(Hint: use your result from V.1.2.)

(c) Suppose that $X = C \times C'$, where C and C' are two curves of genus g and g'. Show that $K^2 = 8(g - 1)(g' - 1)$.
(Hint: write $K_x = p_1^* K_C + p + 2^* K_{C'}$.)