MATH 595 Tuesday 24 April

Numerical equivalence for divisors on surfaces

(1) **Exercise V.1.6**

(a) If C is a smooth curve of genus g, prove that the diagonal $\Delta \subset C \times C$ has self-intersection number $\Delta^2 = 2 - 2g$.
(Hints: use the method of calculating intersection numbers using the degree of a line bundle on a curve; also use the definition of $\Omega_{C/k}$ as $\mathscr{I}/\mathscr{I}^2$, and the fact that this is a line bundle on C of degree $2g-2$.)

(b) Let $\ell = C \times \mathrm{pt}$ and let $m = \mathrm{pt} \times C$. Assume that $g \geq 1$. Show that ℓ, m, and Δ are linearly independent in $\mathrm{Num}(C \times C)$ by proving that if

$$(a\ell + bm + c\Delta).D = 0$$

for all divisors D on $C \times C$, then $a = b = c = 0$.

This allows us to conclude that $\mathrm{Num}(C \times C)$ has rank at least 3. In particular, $\mathrm{Pic}(C \times C)$ is not isomorphic to $p_1^*\mathrm{Pic}C \oplus p_2^*\mathrm{Pic}C$.

(2) **Exercise V.1.7** *Algebraic equivalence of divisors* Let X be a surface, and let T be a non-singular curve. An *algebraic family of effective divisors on* X parametrized by T is an effective Cartier divisor D on $X \times T$ flat over T.

Given such a family D, and any two closed points $0, 1 \in T$, we say that the corresponding divisors D_0, D_1 are *pre-algebraically equivalent*.

Two arbitrary divisors D and D' are *pre-algebraically equivalent* if we can write $D \sim E - F$, $D' \sim E' - F'$ where (E, E') and (F, F') are pairs of pre-algebraically equivalent effective divisors.

Finally, two divisors D and D' are *algebraically equivalent* if there is a finite sequence

$$D = D_0, D_1, \ldots, D_n = D'$$

of divisors such that for all $i = 0, \ldots n-1$, D_i and D_{i+1} are pre-algebraically equivalent.

(a) (Optional.) Show that the set of divisors algebraically equivalent to 0 forms a subgroup.

(b) Show that linearly equivalent divisors are algebraically equivalent, by showing that any principal divisor is algebraically equivalent to 0.
(Hint: if (f) is a principal divisor on X, consider $T = \mathbb{P}^1$ with homogeneous coordinates x, u, and consider the principal divisor $(tf - u)$ on $X \times \mathbb{P}^1$.)

(c) Show that D, D' are algebraically equivalent, and H is very ample, then $D.H = D'.H$.
(Hint: recall that the degree of fibres in a subscheme $Z \subset \mathbb{P}^N_T$ flat over T is constant as we move along T.)

(d) Conclude that if D, D' are algebraically equivalent, then they are also numerically equivalent.