Last time: A plane in $\mathbb{R}^3$ is determined by specifying
- a point $P_0 = (x_0, y_0, z_0)$
- a normal vector $\vec{n} = \langle a, b, c \rangle$.

The plane has equation $ax + by + cz + d = 0$, where d is a constant. $\boxed{2}$

- Take two planes in $\mathbb{R}^3$ and intersect them. $\boxed{1}$

- Suppose $\mathbb{P}_1$ and $\mathbb{P}_2$ are two planes which intersect forming a line L.
  - if we can find two points PQ in L, we can write down the parametric equation of the line $\overleftrightarrow{PQ}$.

  OR - if we can find one point P, and we know the direction of the line, we can write down an equation for L.

  ↳ the line will be orthogonal to $\vec{n}_1$ and $\vec{n}_2$

  so we need to find a vector $\vec{v}$ with $\vec{v} \cdot \vec{n}_1 = 0$, $\vec{v} \cdot \vec{n}_2 = 0$.

Today: Cross products (§12.4)

GOAL: Given $\vec{a}, \vec{b}$ non-zero vectors find $\vec{c} = \langle c_1, c_2, c_3 \rangle$ orthogonal to both of them.

i.e. find a non-zero solution to

① $\vec{a} \cdot \vec{c} = a_1 c_1 + a_2 c_2 + a_3 c_3 = 0$

② $\vec{b} \cdot \vec{c} = b_1 c_1 + b_2 c_2 + b_3 c_3 = 0$.

↳ can take
$$\left.\begin{aligned} c_1 &= a_2 b_3 - a_3 b_2 \\ c_2 &= a_3 b_1 - \cancel{b_3} a_1 b_3 \\ c_3 &= a_1 b_2 - a_2 b_1 \end{aligned}\right\}$$

Definition: $\vec{c} = \langle c_1, c_2, c_3 \rangle$ is the cross product / vector product of $\vec{a}$ and $\vec{b}$.

Notation: $\vec{a} \times \vec{b}$.

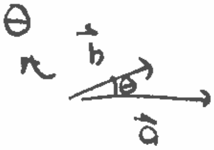
<u>Check:</u> $\vec{a}\cdot(\vec{a}\times\vec{b}) = a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1)$

$= 0.$

Likewise, $\vec{b}\cdot(\vec{a}\times\vec{b}) = 0.$

<u>Geometric characterization:</u> $\vec{a}\times\vec{b}$ is the unique vector which is

- orthogonal to $\vec{a}$ and $\vec{b}$ ✓
- with direction determined by the Right Hand Rule ①
- with length given by $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$ ②

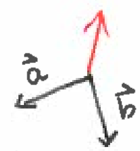


① <u>Right hand rule:</u>

fingers $\vec{a}$

palm $\vec{b}$

thumb $\vec{a}\times\vec{b}$.



$\boxed{i}$

~~↳ Note: from this we can see $\vec{b}\times\vec{a} \neq \vec{a}\times\vec{b}$~~ (unless they're 0)

② <u>Theorem:</u> $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta \qquad (0 \le \theta \le \pi)$

<u>Why?</u> $|\vec{a}\times\vec{b}|^2 = \ldots$

$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$

$$= |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$$

$$= |\vec{a}|^2|\vec{b}|^2 - (|\vec{a}|\cdot|\vec{b}|\cos\theta)^2$$

$$= |\vec{a}|^2|\vec{b}|^2\,\underbrace{(1-\cos^2\theta)}_{\sin^2\theta}$$

$\Rightarrow$ (since $\sin^2\theta > 0$)

$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta.$$

☺

↳ <u>Note:</u> from this we can see $\vec{b}\times\vec{a} = -\vec{a}\times\vec{b}$.

How are we supposed to remember this formula?

<u>Definition:</u> the determinant of a 2x2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

[e.g. $\begin{vmatrix} 2 & 3 \\ -1 & 0 \end{vmatrix} = 2(0) - 3(-1) = 0 + 3 = 3.$]

<u>Definition:</u> the determinant of a 3x3 matrix is:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} := a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

↑ don't forget this sign!

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

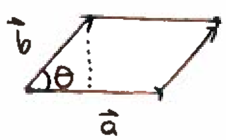
<u>Memory device:</u> $\vec{b} \times \vec{c} = \vec{i}(b_2c_3 - b_3c_2) + \vec{j}(b_3c_1 - b_1c_3) + \vec{k}(b_1c_2 - b_2c_1)$

$$= \text{"}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\text{"}$$

<u>Example</u> $\boxed{i}$

<u>Geometric interpretation of the theorem.</u>

- $\vec{a}, \vec{b}$ determine a parallelogram.



Base $= \vec{a}$

Height $= |\vec{b}| \sin\theta$

$\Rightarrow$ area $A = |\vec{a}||\vec{b}|\sin\theta = |\vec{a} \times \vec{b}|$

<u>Fact.</u> Take $\vec{a}, \vec{b} \neq 0$. Then they are parallel

$\Leftrightarrow$ ~~Ang~~ $\theta = 0,$ ~~or~~ π

$\Leftrightarrow \sin\theta = 0$

$\Leftrightarrow A = |\vec{a} \times \vec{b}| = 0$

$\Leftrightarrow \vec{a} \times \vec{b} = 0.$

<u>Properties of the cross product:</u> [slide]

1) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

2) $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$

3) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

4) $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

5) $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

6) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

WARNING:

- $\vec{a}\times\vec{b} \neq \vec{b}\times\vec{a}$ "not commutative"
- $\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$ "not associative"

proof of (5): $\vec{a}\cdot(\vec{b}\times\vec{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$

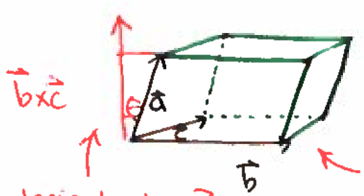
$= (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3$

$= (\vec{a}\times\vec{b})\cdot\vec{c}$.

Rmk: $\vec{a}\cdot(\vec{b}\times\vec{c})$ is called the scalar triple product.

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Geometric significance of the scalar triple product:



parallelepiped.

height $h = ?$

base has area $A = |\vec{b}\times\vec{c}|$

θ = angle made by $\vec{b}\times\vec{c}$ and $\vec{a}$

$\Rightarrow h = |\vec{a}||\cos\theta|$

$\Rightarrow$ volume $= Ah = |\vec{a}||\vec{b}\times\vec{c}|\cos\theta$

$\boxed{= |\vec{a}\cdot(\vec{b}\times\vec{c})|}$.

Volume of parallelepiped.

*note: if it's 0, $\vec{a},\vec{b},\vec{c}$ are coplanar.

Example of intersecting planes:

Two planes $x + y + z = 1$ $\vec{n}_1 = \langle 1,1,1\rangle$

$x - 2y + 3z = 1$ $\vec{n}_2 = \langle 1,-2,3\rangle$

Line L of intersection will be orthogonal to $\vec{n}_1, \vec{n}_2$

i.e. paral in direction $\vec{n}_1\times\vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = \vec{i}5 - \vec{j}2 - \vec{k}3$

$= \langle 5,-2,-3\rangle$.

- Find a point in L:

Set $z = 0$. $\left.\begin{array}{l} x + y = 1 \\ x - 2y = 1 \end{array}\right\} \Rightarrow y = 0,\ x = 1$

$\Rightarrow P = (1,0,0) \in L.$ $\boxed{1}$