Equation of a plane in $\mathbb{R}^3$.

The plane which contains the point $P_0 = (x_0, y_0, z_0)$ and has normal vector $\mathbf{n} = \langle a, b, c \rangle$ is given by the equation

$$ax + by + cz + d = 0.$$

Here d is the constant number given by

(a) $d = ax_0 + by_0 + cz_0$;

(b) $d = -ax_0 - by_0 - cz_0$;

(c) $d = x_0^2 + y_0^2 + z_0^2$;

(d) I don't know.

Correct answer: (b)

Intersection of two planes in $\mathbb{R}^3$.

Take two planes in $\mathbb{R}^3$ and intersect them. The set of point in the intersection could form

(a) a single point;

(b) a line;

(c) there could be no points in the intersection;

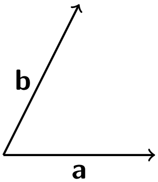
(d) either (b) or (c) could happen.

Case (c) happens $\iff$ the planes are parallel $\iff$ the normal vectors are parallel.

Case (b) happens whenever they aren't parallel. Then we want to determine the equation of this line.

The Right-Hand Rule

Consider the following vectors:



Then $\mathbf{a} \times \mathbf{b}$ points

(a) into the board;

(b) out of the board.

Correct answer: (b)
Note that $\mathbf{b} \times \mathbf{a}$ points into the board.

Cross product: example

Take $\mathbf{a} = \langle 1, 0, 1 \rangle$, $\mathbf{b} = \langle 4, 2, 0 \rangle$.

Then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 4 & 2 & 0 \end{vmatrix} = ?$$

(a) $-2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$;

(b) $2\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$;

(c) $-2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$;

(d) I don't know.

Correct answer: (c)

Properties of the cross product

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$;
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$;
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$;
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$;
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$;
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

Two intersecting planes in $\mathbb{R}^3$

Take two planes with equations

$$\begin{aligned} x + y + z &= 1; \\ x - 2y + 3z &= 1. \end{aligned}$$

They intersect forming a line L.

This line will be perpendicular to the normal vectors $\mathbf{n}_1 = \langle 1, 1, 1 \rangle; \mathbf{n}_2 = \langle 1, -2, 3 \rangle$, so its direction is the same as that of

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = \langle 5, -2, -3 \rangle.$$

We can also find a point P in L as follows: let's look for a point with $z = 0$.

Then the equations become

$$\begin{aligned} x + y &= 1; \\ x - 2y &= 1. \end{aligned}$$

From this we see that $y = 0, x = 1$, and so $P = (1, 0, 0) \in L$.

Recall that we already showed that L has direction $\langle 5, -2, -3 \rangle$.

Which of the following gives an equation for L?

(a) $\mathbf{r}(t) = \langle \frac{1+t}{5}, \frac{t}{-2}, \frac{t}{-3} \rangle$;

(b) $\mathbf{r}(t) = \langle 1 + 5t, -2t, -3t \rangle$;

(c) I don't know.

Correct answer: (b)