[1] Intersecting planes in $\mathbb{R}^3$.

Announcement: Math help room is in AH147, M–R.
Check website for updates, as it may be moved again.

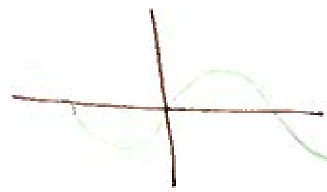
**Today:** VISUALIZING FUNCTIONS OF SEVERAL VARIABLES.

What you already know:

Given a function $f: \mathbb{R} \to \mathbb{R}$, it is helpful to draw its graph

$$\Gamma(f) = \{(x, f(x)) \mid x \in \mathbb{R}\} \subset \mathbb{R}^2.$$

e.g. $f(x) = \sin x$



Functions in higher dimensions.

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ ; or maybe $D \subset \mathbb{R}^n$.

↑ "maps from" ↖ "to"

$f: D \to \mathbb{R}^m$ ↑ domain

$(x_1, \ldots, x_n) \longmapsto f(x_1, \ldots, x_n)$ ↖ "maps to"

Examples:

1) Temperature $T$ in this room.

Let $A = \{(x, y, z) \in \mathbb{R}^3 \mid \text{a point in AH314}\} \subset \mathbb{R}^3$.

$T: A \to \mathbb{R}$

$P = (x, y, z) \longmapsto T(P) = \text{temperature at time } T.$ (in °C)

Note: in general, not all values of $\mathbb{R}$ are equal to $T(P)$ for some $P$.

Those values which are are in the range of $T$.

2) A Location $L$ of a fly in the room, measured throughout class:

$L: [1{:}00 - 1{:}50] \longrightarrow A$.

$t \longmapsto L(t) = \text{location at time } t.$

$= (x(t), y(t), z(t)).$

Domain: set of input values on which the function is defined.

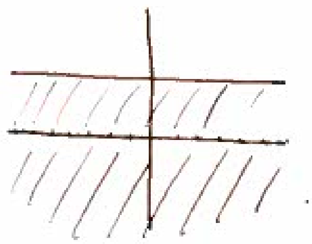
Sometimes you'll be given an algebraic formula and D won't be specified, so you have to figure it out.

e.g. $f(x,y) = x\sqrt{1-\frac{1}{y}}$ ← not defined if $y=0$

or if $1-\frac{1}{y} < 0$

i.e. if $1 < \frac{1}{y}$

i.e. if $y > 1$.



$\therefore D = \{(x,y) \mid y \neq 0, y \leq 1\}$

[i]

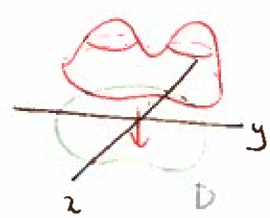
§. THE GRAPH OF A FUNCTION OF TWO VARIABLES.

$f: D \to \mathbb{R}$

$D \subset \mathbb{R}^2$.

$\Gamma(f) = \{(x,y,f(x,y)) \in \mathbb{R}^3 \mid (x,y) \in D\}$.

↳ it will be a surface.

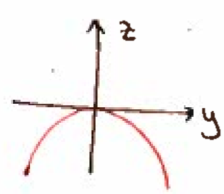



It might be hard to draw.

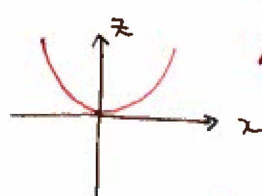
<u>Example</u> $z = x^2 - y^2$.

take traces/intercepts by intersecting with different planes,

e.g. • $x = 0 \Rightarrow z = -y^2$ ← parabola

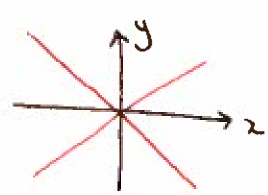



• $y = 0 \Rightarrow z = x^2$




• $z = 0 \Rightarrow x^2 - y^2 = 0$

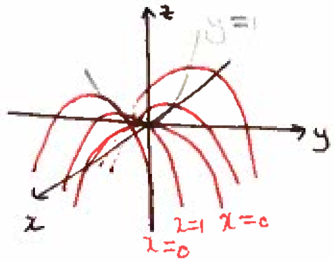
$\Rightarrow x = \pm y$




• $x = k$: $z = k^2 - y^2$

• $y = k$ $z = x^2 - k^2$.

## Putting the slices together:




⇝ saddle:

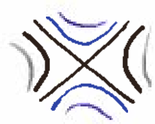
"hyperbolic paraboloid"

**Def** the slices $z = k$ are called "level sets" – intersection of $\Gamma(f)$ with the plane $z = k$.

$\{(x,y) \mid f(x,y) = k\}$.

**Example:** $x^2 - y^2 = z$ again

$z = 0$:



as before

$z = 1$

$x^2 - y^2 = 1$

$\Rightarrow x = \pm\sqrt{1+y^2}$

$z = 2$

$x^2 - y^2 = 2$

$x = \pm\sqrt{2+y^2}$

$z = -1$

$x^2 - y^2 = -1$

$\Rightarrow y = \pm\sqrt{1+x^2}$

$z = -2$

$y = \pm\sqrt{2+x^2}$

**Definition:** this diagram is called the contour map (like a topographical map of mountains)

- even though it can be drawn in 2d, it still helps us visualize the function.

[i] **Question:** Draw the contour map and the graph of $f(x,y) = x^2 + y^2$.

[ii] **Question:** Look at contour lines for evenly spaced k.

# § FUNCTIONS IN THREE VARIABLES.

Example: $f(x,y,z) = x^2 + y^2 + z^2$.

the graph will live in $\mathbb{R}^4$!

$\Rightarrow$ we can't draw it.

But: each level set lives in $\mathbb{R}^3$:

$\{(x,y,z) \mid f(x,y,z) = k\}$

So we can draw that

eg

e.g. · $x^2 + y^2 + z^2 = 0 \iff (x,y,z) = (0,0,0)$

· if $k < 0$,

$x^2 + y^2 + z^2 = k$ has no solutions.

· if $k > 0$.

$x^2 + y^2 + z^2 = k$ ← sphere of radius $\sqrt{k}$.

← contour map.

Example: "Hopf fibration"

$S^3 = \{(x,y,z,w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\}$

$H: S^3 \to S^2$

$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$.

Fix $P \in S^3$.

Level set of $P$ is a circle.

(See video)