MATH 241 - LECTURE 14 - ABSOLUTE MAX/MIN §14.7 - Monday 18 Feb. 2019

Last time: Second derivative test.

Consider the function $f(x,y) = x^2 - 2xy + 2y$.

[1] Classify its critical points

- Second derivative test for $g(x,y) = ax^2 + bxy + cy^2$

[See slides]

§ ABSOLUTE MAXIMA / MINIMA.

One variable: $f: [p,q] \to \mathbb{R}$ continuous

$\Rightarrow$ f ~~attains~~ attains an absolute maximum value at some point $a \in [p,q]$

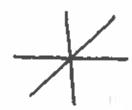
- a is either a boundary point (p, q)
  or a critical point.

Likewise for absolute min.

Warnings: False for:

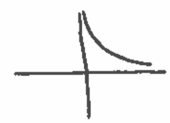
- $f: \mathbb{R} \to \mathbb{R}$

  e.g. $f(x) = x$



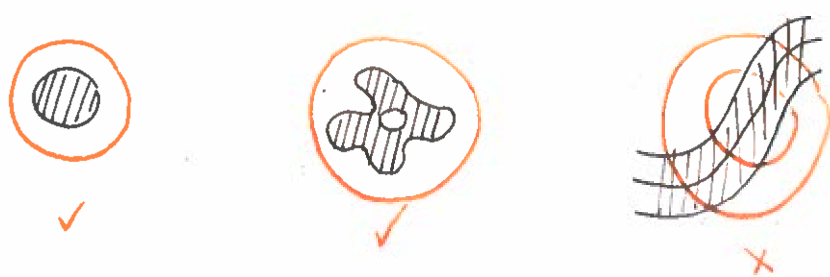
- $f: (p,q) \to \mathbb{R}$

  e.g. $f: (0,1] \to \mathbb{R}$
  $x \mapsto \frac{1}{x}$

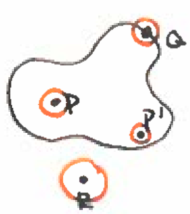


- f discontinuous.

Definition $D \subset \mathbb{R}^2$ is bounded if it is contained in some disk.



Definition $(a,b) \in \mathbb{R}^2$ is a boundary point of D if every ~~point~~ disk around (a,b) contains points in D AND points not in D.



P - not a boundary point

P' - still not a boundary point

R - not a boundary point

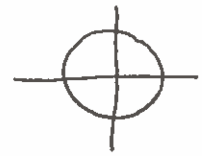
Q - is a boundary point!

• Similar definitions for $D \subset \mathbb{R}^3$.

[Q]. How many ~~boundary~~ points are in the boundary of $[p,q] \subset \mathbb{R}$?

<u>Definition</u>: D is closed if it contains all its boundary points.

<u>Example</u>: 1) $D = \{(x,y) \mid x^2+y^2 \leq 1\}$

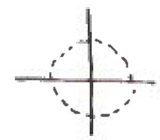


Boundary $= \{(x,y) \mid x^2+y^2 = 1\}$

$\subseteq D$.

$\Rightarrow$ D is closed

2) $D = \{(x,y) \mid x^2+y^2 < 1\}$



Boundary is NOT in D

$\Rightarrow$ D is not closed.

3) $[p,q]$, $\mathbb{R}$ - closed

$[p,q)$, (p,q) - not closed.

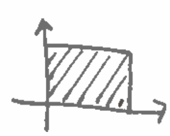
<u>Note</u>: in practice, if D is determined by conditions like

• $g(x,y) \leq 0$, $h(x,y) \geq 0$, $f(x,y) = 0$

it will be closed.

But not by ~~equations~~ conditions like $g(x,y) > 0$, $h(x,y) < 0$

[Q] Let $D = \left\{ \begin{matrix} 0 \leq x \leq 3 \\ 0 \leq y \leq 2 \end{matrix} \right\}$.



Is D closed? Is D bounded?

[D closed and bounded is the analogue to the interval $[p,q]$]

Suppose we have $D \subset \mathbb{R}^2$ and $f: D \to \mathbb{R}$.

<u>Definition</u> f has an {absolute maximum / absolute minimum} at (a,b) if $f(x,y) \geq f(a,b)$ / $f(x,y) \leq f(a,b)$

for all $(x,y) \in D$.

Theorem [Extreme value theorem]

Suppose $f: D \to \mathbb{R}$ is continuous and D is closed and bounded.

Then f achieves an absolute maximum value at some point (a,b) in D,

and (a,b) is either
- on the boundary of D
- OR a critical point of f.

(Likewise for absolute min).

Strategy for finding absolute max:

(1) Find all critical points and evaluate f at each.

(2) Find all boundary points and find the maximum of f there

(3) Take the maximum of the values in (1) and (2).

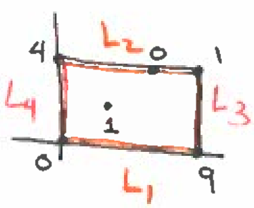
Example Find the absolute max/min of $f(x,y) = x^2 - 2xy + 2y$

on $D = \left\{ \begin{matrix} 0 \le x \le 3 \\ 0 \le y \le 2 \end{matrix} \right\}$.

Note: EVT applies: f is continuous

D is closed and bounded.

(1) Critical point: $(1,1)$. $f(1,1) = 1 - 2 + 2 = 1$



(2) Boundary: divided into four line segments

On L_1: $g(x) = f(x,0)$, $0 \le x \le 3$

$= x^2$

⇒ local min at $x = 0$.

$g(0) = 0$

absolute max at $x = 3$

$g(3) = 9$

On L_3: $g(x) = f(x,2) = x^2 - 4x + 4. = (x-2)^2$ on $0 \le x \le 3$.

⇒ min at $x = 2$, $g(2) = 0$

⇒ max at $x = 0$, $g(0) = 4$

(check also $x = 3$: $g(3) = 1$

Note: on L_3, $g(y) = f(3,y) = 9 - 6y + 2y = 9 - 4y$.

extrema are the endpoints we already found.

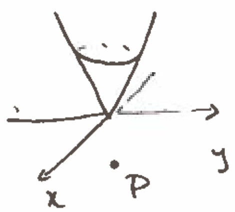
Likewise on L_4, $g(y) = f(0,y) = 2y$

Step 3. Abs. max: $f(0,3) = 9$

Abs. min: $f(0,0) = f(2,2) = 0$. }.

Another example

Find the points on the graph $z = \sqrt{x^2+y^2}$ that are closest to/furthest from the point $(4,2,0)$.



Note: there should be a closest point, but not a furthest point, because we can get as far from $(4,1,0)$ as we like.

Consider the distance squared from $(x,y,\sqrt{x^2+y^2})$ to $(4,1,0)$:

$$f(x,y) = (x-4)^2 + (y-2)^2 + (\sqrt{x^2+y^2} - 0)^2$$

$$= (x^2 - 8x + 16) + (y^2 - 4y + 4) + (x^2+y^2)$$

$$= 2x^2 + 2y^2 - 8x - 4y + 20.$$

Critical points:

$f_x(x,y) = 4x - 8 = 0 \Rightarrow x = 2$

$f_y(x,y) = 4y - 4 = 0 \Rightarrow y = 1$

} only $(x,y) = (1,2)$.

$\Rightarrow$ f has local minimum and $(1,2)$.

$\Rightarrow$ closest point is $Q = (1,2\sqrt{5})$.