Last time: space curves and arc-length

Recall the formula

$$L = \int_a^b |\mathbf{r}'(t)| dt.$$

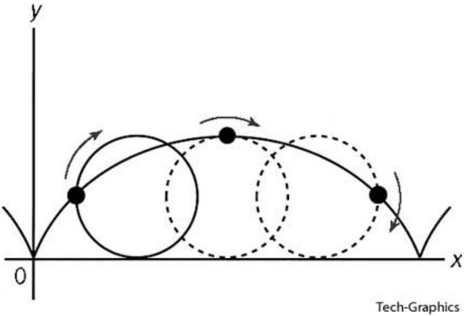
Use this to calculate the length of the curve with equation

$$x = \frac{2}{3}(y-1)^{\frac{3}{2}}, 1 \leq y \leq 4.$$

(a) 0
(b) $\frac{14}{3}$
(c) $\frac{18}{3}$
(d) I don't know how to find a parametrization $\mathbf{r}(t)$.
(e) I found a parametrization $\mathbf{r}(t)$, but I can't integrate the result.

Correct answer: (b)

Cycloid



Tautochrone *("same time")*

Brachistochrone *("shortest time")*

Integration in one variable

Fix $g : [a, b] \to \mathbb{R}$.

- Divide $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal length $\Delta x = \frac{b-a}{n}$.
- For each i choose any $x_i^* \in [x_{i-1}, x_i]$.

$$\int_a^b g(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i^*)\Delta x$$

(assuming the limit exists and is independent of the choices of x_i^*).

Geometric interpretation of integration

- $\int_a^b g(x)dx = (b-a) \times$ (average value of g on $[a, b]$).
- If $g \geq 0$, $\int_a^b g(x)dx$ gives the area under the graph of g over the interval $[a, b]$.
- If $g(x)$ is the linear density of a straight piece of wire with endpoints a and b, then $\int_a^b g(x)dx$ calculates the total mass of the wire.
- Furthermore, the point $\overline{x} \in [a, b]$ corresponding to the centre of mass of the wire is given by

$$\overline{x} = \frac{\int_a^b xg(x)dx}{\int_a^b g(x)dx}$$

Practice with integration

Let C be the semicircle given by $x^2 + y^2 = 1$, $y \geq 0$, and consider $f(x, y) = y$.

Calculate $\int_C f ds$, using the parametrization $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, $t \in [0, \pi]$.

(a) 1

(b) -1

(c) 0

(d) 2

(e) I don't know how.

Correct answer: (d)

Studying a wire

Suppose we have a wire bent in the shape of the semicircle C given by $x^2 + y^2 = 1$, $y \geq 0$. Assume the wire has constant linear density ρ.

Use geometric intuition to guess the centre of mass from the options below.

(a) $(0, 0)$
(b) $(0, 1)$
(c) $(1, 1)$
(d) $(\frac{2}{\pi}, \frac{2}{\pi})$
(e) $(0, \frac{2}{\pi})$

Correct answer: (a)