MATH 241 - LECTURE 18 - VECTOR FIELDS (§§ 16.1, 16.2) - Weds. 27 Feb. 2019

Last time: Integrating functions over curves.

$\boxed{2}$ Compute $\int_C x^2 z\,ds$ where C is the line segment from $(0,6,-1)$ to $(4,1,5)$.

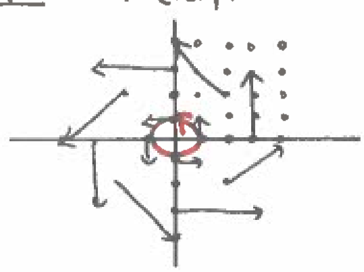
§ VECTOR FIELDS (§16.1)

Definition: A **vector field** on $\mathbb{R}^2$ is a function assigning to each point $(x,y) \in \mathbb{R}^2$ a vector $\vec{F}(x,y) = \langle P, Q \rangle$

Notation: $\vec{F} : \mathbb{R}^2 \to V_2$ or $\mathbb{R}^2 \to \mathbb{R}^2$

To visualize, draw $\vec{F}(x,y)$ with tail (x,y).
(rescale arrows if needed).

Example. $\vec{F}(x,y) = \langle -y, x \rangle$



Definition: A vector function $\vec{r}(t)$ is called a **flow of F** if $\vec{F}(\vec{r}(t)) = \vec{r}\,'(t)$ for all t.

- in this case if $\vec{r}(t) = \langle \lambda \cos t, \lambda \sin t \rangle$,

$$\vec{F}(\vec{r}(t)) = \langle -\lambda \sin t, \lambda \cos t \rangle = \vec{r}\,'(t).$$

Example: Given any ~~continuous~~ $f : \mathbb{R}^2 \to \mathbb{R}$, we define its **gradient vector field** $\nabla f : \mathbb{R}^2 \to V_2$

$$(x,y) \longmapsto \nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle.$$

- if a vector field F can be expressed as ∇f for some f, we say that F is **conservative** and f is a **potential function/potential energy** for F.

Similar definitions for a **vector field on $\mathbb{R}^3$**

Example: Gravitational field.

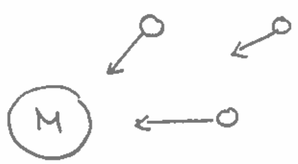
A large mass M at $(0,0,0)$ exerts a gravitational force $\vec{F}(x,y,z)$ on a small mass m at position $\vec{r} = \langle x,y,z\rangle$.

- direction of force: $-\vec{r}$.
- Newton's law: $|\vec{F}| = \dfrac{MmG}{|\vec{r}|^2}$ ← Gravitational constant
- $\vec{F} = -c\vec{r} \qquad (c \geq 0)$

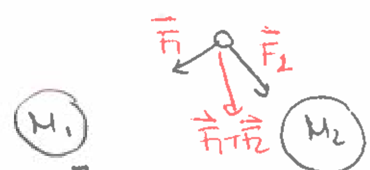
$$\Rightarrow |\vec{F}| = c|\vec{r}| = \frac{MmG}{|\vec{r}|^2} \quad \Rightarrow \quad c = \frac{MmG}{|\vec{r}|^3}$$

$$\therefore \vec{F} = -\frac{MmG}{|\vec{r}|^3}\vec{r}$$

i.e. $$\vec{F}(x,y,z) = -\frac{MmG}{(x^2+y^2+z^2)^{3/2}}\langle x,y,z\rangle$$



For several bodies, we can add vector fields.



[Another example. Electric field generated by a charged particle.]

Remark: The gravitational field is conservative:

Take $g: \mathbb{R}^3 \to \mathbb{R}$

$$(x,y,z) \longmapsto g(x,y,z) = \frac{MmG}{\sqrt{x^2+y^2+z^2}}$$

$$\nabla g(x,y,z) = \langle MmG\left(-\tfrac{1}{2}(x^2+y^2+z^2)^{-3/2}\, 2x\right), \ldots \rangle$$

$$= \left\langle -\frac{MmG}{(x^2+y^2+z^2)^{3/2}}x,\ \frac{-MmG}{(x^2+y^2+z^2)^{3/2}}y,\ \frac{-MmG}{(x^2+y^2+z^2)^{3/2}}z \right\rangle$$

$$= \vec{F}(x,y,z).$$

§ 16.2. LINE INTEGRALS OF VECTOR FIELDS.

Fix: · a smooth plane curve C with parametrization $\vec{r}(t) = \langle x(t), y(t) \rangle$, $t \in [a,b]$.

· $\vec{F} = \langle P, Q \rangle$ - continuous vector field on C

$P, Q : C \to \mathbb{R}$, "components" of F.

<u>Definition</u>: the ~~line~~ **line integral of F along C** is

$$\int_C \vec{F} \cdot d\vec{r} := \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \langle P(\vec{r}(t)), Q(\vec{r}(t)) \rangle \cdot \langle x', y' \rangle \, dt$$

$$= \int_a^b \left(P(\vec{r}(t))\, x'(t) + Q(\vec{r}(t))\, y'(t) \right) dt$$

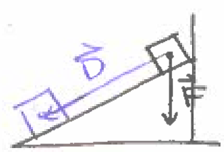
<u>Notation</u>: we also write $\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy$.

· Similar formula for C in $\mathbb{R}^3$.

<u>Physical interpretation</u>: Work.

· Recall that for a constant force $\vec{F}$ displacing an object by $\vec{D}$.

the **work** is $W = \vec{F} \cdot \vec{D}$



(here, positive work).

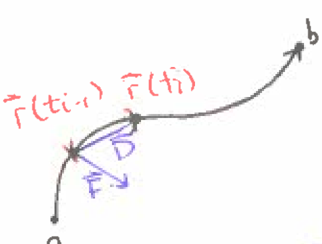
But what if the force is not constant?

Let $\vec{r}(t)$, $t \in [a,b]$ denote the position of a rocket

As the rocket moves, the gravitational force on it changes:

$$\vec{F}(\vec{r}(t)) = \frac{-\cancel{G}\cdot MmG}{|\vec{r}|^3}\, \vec{r}(t).$$

What is the total work done as the rocket moves from $\vec{r}(a)$ to $\vec{r}(b)$?



divide the path into segments.

Work done on one segment is approximated:

$$\approx \vec{F}(\vec{r}(t_{i-1})) \cdot (\underbrace{\vec{r}(t_i) - \vec{r}(t_{i-1})}_{\approx \Delta t\, \vec{r}\,'(t_{i-1})})$$

$$\approx \left(\vec{F}(\vec{r}(t_{i-1})) \cdot \vec{r}\,'(t_{i-1}) \right) \Delta t.$$

Sum up and take $n \to \infty$, $\Delta t \to 0$:

Total work $W = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt$

$= \int_C \vec{F} \cdot d\vec{r}.$

$\boxed{1}$ <u>Example</u>: Let $\vec{r}(t) = \langle t, t^2 \rangle$, $\vec{F}(x,y) = \langle y, x \rangle$

$t \in [0,1]$.

Sketch the curve and vector field. Estimate.

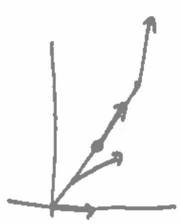
. Work out the details.

(See "solution" i-clicker slides)

<u>Another example:</u>

$\boxed{2}$ Let C be parametrized by $\vec{r}(t) = \langle t, 2t \rangle$, $t \in [0,1]$

Let $\vec{F}(x,y) = \langle 1, 2y \rangle$. Find $\int_C \vec{F} \cdot d\vec{r}$



- $\cancel{\vec{F}(x,y) =}$
- $\vec{F}(\vec{r}(t)) = \langle 1, 4t \rangle$
- $\vec{r}\,'(t) = \langle 1, 2 \rangle$

$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \int_0^1 \langle 1, 4t \rangle \cdot \langle 1, 2 \rangle\ dt$

$= \int_0^1 (1 + 8t)\,dt$

$= \left[t + 4t^2\right]_0^1$

$= 5.$